

Research Article

The effect of natural convection induced thermal explosion in a cylindrical reactor with temperature-dependent thermal conductivity

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Abstract

This paper examines the stability and effects of thermal explosions in porous, cylindrical reactors that experience natural convection and have temperature-dependent thermal conductivity. The momentum and energy equations are constructed using the Darcy-Brinkman theory and the Boussinesq hypothesis. Using Lie group's symmetry, the energy and momentum equations are translated into ordinary differential equations and are then solved using the regular perturbation approach. The parameter defining the critical region for thermal explosions, the Frank-Kamenetskii parameter, is calculated. A linear stability analysis is carried out to evaluate the ignition threshold. The details of stability boundary effects of different values of the thermal conductivity and Darcy number of the porous medium are thoroughly studied. It is demonstrated that stability margins are enlarged for high values of the thermal conductivity parameter and the system is unstable for higher Darcy numbers. All the analytical results were found to be in observational agreement in different system limits, further confirmed by TikZ simulations.

Keywords: Thermal explosion; Natural convection; Thermal conductivity; Frank-Kamenetskii parameter.

1. Introduction

Thermal explosions remain a major safety concern in the processing and chemical manufacturing of products where exothermic reactions occur in confined geometries (Ahmed et al., 2024; Kumar and Singh, 2018). Since the introduction of the theories by Semenov and Frank-Kamenetskii in the early 1900s, there have been a number of advancements in the understanding of the phenomena of ignition in idealized homogeneous scenarios (Frank-Kamenetskii, 2023; Kumar et al., 2022; Zhang and Liu, 2023). However, the more recent industrial reactors like underground coal gasification, biomass gasifiers, some nuclear waste repositories, and the more traditional packed-bed catalytic converters do not adhere to the idealized homogeneous or simple scenarios of fluidized and saturated porous media with natural convection and temperature dependent properties of matter among other things (Singh and Das, 2023; Zhang and Liu, 2023). Despite extensive research, a significant gap in the literature remains which is concerning and that is the lack of understanding and characterization of the phenomenon of thermal explosion as a result of the coupling of natural convection and temperature dependent thermal conductivity in the cylindrical porous reactors (Li et al., 2023; Wang and Zhao,

2023). There are multiple reasons why such gaps are concerning. For instance, natural convection may completely destabilize or stabilize a given reacting system depending on which flow regime it settles in. This may result in the advection of the warmer reaction fluid to the cooler parts of a region thus thermal runaway may be locally advanced, or it may be removed from the reaction zone, thus postponing ignition (Rahman et al., 2024; Singh et al., 2023; Zhang et al., 2015). Also, there is a significant temperature dependence to the effective thermal conductivity of most of the porous materials which varies by 20-50% (Chen et al., 2024; Wang and Xu, 2017).

Recent numerical studies demonstrate that based on this assumption, errors greater than 30% may be expected when estimating the critical parameters for ignition (Patel et al., 2024; Zhang et al., 2024).

The prediction of stability is highly dependent on the choice of a transport model for momentum. The classical law of Darcy is still popular, however, it entirely omits the effect of viscous shear, which is important in the vicinity of solid boundaries, particularly in small diameter reactors and in media of high permeability (Das and Sen, 2023; Nield and

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Received: 24/03/2026; Accepted: 04/06/2026; Published: 31/07/2026



Bejan, 2013). The Darcy-Brinkman model incorporates both viscous shear and bulk frictional resistance, and predicts more realistic velocity distributions and therefore enhanced predictions of heat transfer (Kumar et al., 2024; Xu et al., 2020; Vadasz, 2008).

These phenomena can be modeled by a series of coupled nonlinear phenomena, each with its own mathematical challenge. The governing PDEs will have nonlinearities from the Arrhenius reaction, buoyancy driven flows and a change in thermal conductivity (Mahmoudi and Karimi, 2014; Nield and Bejan, 2013). While the traditional numerical techniques are valuable, they tend to obscure the parametric relationship that is governing the stability limits, and give little analytical insight on the critical conditions that define the boundaries between safe and thermal runaway operation (Islam et al., 2024; Nayfeh and Pai, 2024). In response to these challenges, this study develops a systematic analytical framework that integrates two mathematical techniques. Lie group symmetry analysis (Ibragimov, 1999; Olver, 1993) is employed to reduce the governing PDEs to an ordinary differential equation (ODE) by identifying invariant transformations that preserves the essential physics of the system.

This approach has recently gained traction in porous media convection problems due to its ability to uncover similarity solutions that numerical methods alone cannot easily reveal (Jena & Mahapatra, 2019; Mohamed and Elsaid, 2024; Rahman and Islam, 2025). Subsequently, a regular perturbation method (Nayfeh, 2008; Nayfeh and Pai, 2024) is applied under some realistic assumptions of small dimensionless parameters γ (thermal conductivity variation) and ϵ (activation energy parameter). This dual approach provides explicit analytical expressions for the critical Frank-Kamenetskii parameter, clearly showing how each physical parameter influences the ignition threshold (Alharbi and Alshomrani, 2021; Hasan and Chowdhury, 2023). The main concern of this study is the following problem statement: What are the possible ways in which a naturally convecting, temperature-dependent, cylindrical porous reactor with exothermic properties transitions from a safe, stable operating regime to an unstable thermal explosion?

This could provide insight into possible industrial design needs, such as:

- (i) designing safer packed-bed reactors for exothermic processes (American Institute of Chemical Engineers, 2024),
- (ii) evaluating the risk of ignition during the carbon sequestration and nuclear waste migration of sediments in clay layers (European Process Safety Centre, 2023),
- (iii) optimizing the design of gasifiers that could thermally run away (and cause damage to the gasifier) (Ali et al., 2024), and

- (iv) improving the process of managing safety to identify thermal hazards early (Kumar and Singh, 2018).

The actual goals of this study include:

1. To formulate the governing equations from the Darcy-Brinkman model with linear temperature-dependent thermal conductivity and appropriate boundary conditions;
2. To derive a similarity transformation via Lie group symmetry analysis that reduces the PDE system to a single ordinary differential equation (ODE) for the steady-state temperature field;
3. To determine analytical expressions for the critical Frank-Kamenetskii parameter, δ , using regular perturbation expansions in the small parameters γ and ϵ [Xu, and Pop, (2020); Zhang, et al., (2024); Zhang, and Li, (2015)]
4. To perform a linear stability analysis that categorizes operating as stable or unstable and maps neutral stability boundaries.
5. To systematically examine the effects of thermal conductivity parameter, activation energy parameter, and Darcy number on the stability of the system, and
6. To validate the analytical predictions against numerical solutions and recover classical limiting cases.

The remainder/summary of this paper is organized as follows. Section 2 presents the mathematical formulation, including the governing equations and dimensionless parameters. Section 3 describes the Lie group symmetry analysis and reduction to an ODE. Section 4 develops the perturbation solutions and the critical Frank-Kamenetskii parameter. Section 5 presents the linear stability analysis. Section 6 discusses the results, including parametric studies and validation. Section 7 concludes the paper with key findings and recommendations for future work.

2. Mathematical Formulation

Consider a porous cylindrical reactor of radius R filled with a fluid-saturated porous medium. The cylinder is subjected to a zero-order exothermic reaction with heat release $-\Delta H$. Fluid motion occurs due to buoyancy forces under the Boussinesq approximation. The thermal conductivity k is assumed to depend linearly on temperature: $k(T) = k_0[1 + \lambda(T - T_0)]$, where λ is the thermal conductivity parameter.

2.1. Governing Equations

The continuity, momentum (Darcy-Brinkman), and energy equations are stated as follows

$$\nabla \cdot \mathbf{u} = 0 \quad (1)$$

$$\frac{\mu}{k} \mathbf{u} = -\nabla p + \rho_0 \beta (T - T_0) g \mathbf{e}_0 + \overline{u \nabla^2} \mathbf{u} \quad (2)$$

$$(\rho c_p) m \frac{\partial T}{\partial t} + (\rho c_p)_f u \cdot \nabla T = \nabla \cdot [k(T) \nabla T + QACe^{-\frac{E}{RT}}] \quad (3)$$

Here, u is the Darcy velocity, T the temperature, p the pressure, μ the viscosity, κ the permeability, β the thermal expansion coefficient, g gravity, μ^* the effective viscosity, Q the heat of reaction, A the pre-exponential factor, C the concentration, E the activation energy, and R the universal gas constant.

2.2. Dimensionless Formulation

By introducing dimensionless variables, we have;

$$r = R\bar{r}, \quad z = R\bar{z}, \quad u = \frac{\alpha}{R}\bar{u}, \quad \theta = \frac{E}{RT_0^2}(T - T_0), \quad (3a)$$

$$t = \frac{R^2}{\alpha}\bar{t}, \quad \alpha = k_0 / (\rho C_p) f$$

Where is the thermal diffusivity? The dimensionless equations become:

$$\nabla \cdot u = 0 \quad (4)$$

$$-\nabla p + Ra\theta e_z + Da\nabla^2 u - u = 0 \quad (5)$$

$$\frac{\partial \theta}{\partial t} + u \cdot \nabla \theta = \nabla \cdot [(1 + \gamma\theta)\nabla \theta] + \delta e^{\frac{\theta}{(1+\epsilon\theta)}} \quad (6)$$

$$Ra = \frac{\rho_0 \beta g k R (E/RT_0^2)}{\mu \alpha}$$

Where Ra is the modified Raleigh number, $Da = \frac{\mu}{\mu} \frac{k}{R^2}$ is the Darcy number, $\gamma = \lambda(RT_0^2/E)$ is the dimensionless thermal conductivity parameter, $\delta = \frac{QACER^2}{k_0 RT^2} e^{-E/(RT_0)}$ is the Frank-Kamenskii parameter, and $\epsilon = RT_0/E$ is the activation energy parameter.

3. Lie Group Symmetry Analysis

Considering Lie group symmetries of the system (4)-(6) in the variables (r, z, t, u, p, θ) , the infinitesimal generator is

$$X = \xi^r \frac{\partial}{\partial r} + \xi^z \frac{\partial}{\partial z} + \xi^t \frac{\partial}{\partial t} + \eta^v \frac{\partial}{\partial v} + \eta^w \frac{\partial}{\partial w} + \eta^p \frac{\partial}{\partial p} + \eta^\theta \frac{\partial}{\partial \theta}$$

Applying the invariance conditions and solving the determining equations yields the following similarity variables and forms:

$$\eta = \frac{r}{f(t)}, \quad \theta = g(t)\Theta(\eta), \quad u = h(t)U(\eta)e_z$$

Where $f(t)$, $g(t)$, $h(t)$ are determined from the symmetry conditions. For steady-state analysis, we set $f(t) = 1$, $g(t) = 1$, $h(t) = 1$, this leads to the reduced ODE system below.

2.2. Reduced Ordinary Differential Equation (RODE)

Under the similarity transformation, the steady-state energy equation (4-6) reduces to:

$$\frac{1}{\eta} \frac{d}{d\eta} [\eta(1 + \gamma\Theta) \frac{d\Theta}{d\eta}] + \delta e^{\Theta/(1+\epsilon\Theta)} = 0 \quad (7)$$

Subject to the boundary conditions this:

$$\frac{d\Theta}{d\eta}(0) = 0, \quad \Theta(1) = 0. \quad (8)$$

4. Perturbation Method

For small γ and ϵ , a regular perturbation expansion is employed:

$$\Theta(\eta) = \Theta_0(\eta) + \gamma\Theta_1(\eta) + \epsilon\Theta_2(\eta) + O(\gamma^2, \epsilon^2, \gamma\epsilon).$$

The reduced problem (by setting $\gamma = \epsilon = 0$) has the same qualitative character as the full problem, and the small parameters multiply terms that do not change the order of the highest derivatives.

For $\gamma = \epsilon = 0$, the system reduces to a pure diffusion problem (or Stokes flow), which has a smooth solution. This can only occur or exist, and be smooth if the solution is unperturbed [Nield, and Bejan, (2013)].

Now, the γ (Thermal Conductivity Parameter) and ϵ is the activation energy parameter) are all small parameters i.e. $\gamma \ll 1$, $\epsilon \ll 1$.

The solution of $\Theta(\eta)$ can be expanded as a regular perturbation series in powers of γ and ϵ as shown below:

$$\Theta(\eta) = \Theta_0(\eta) + \gamma\Theta_1(\eta) + \epsilon\Theta_2(\eta) + \gamma^2\Theta_{11}(\eta) + \gamma\epsilon\Theta_{12}(\eta) + \epsilon^2\Theta_{22} + \dots$$

by keeping the first-order linear terms only (i.e., γ directly proportional to ϵ)

$$\Theta(\eta) = \Theta_0(\eta) + \gamma\Theta_1(\eta) + \varepsilon\Theta_2(\eta) + O(\gamma^2, \varepsilon^2, \gamma\varepsilon)$$

Also, by expanding each terms in the ODE [i. e (1+ $\gamma\Theta$)],

we have below:

$$1 + \gamma\Theta = 1 + \gamma(\Theta_0 + \gamma\Theta_1 + \varepsilon\Theta_2 + \dots) = 1 + \gamma\Theta_0 + O(\gamma^2, \gamma\varepsilon)$$

Now, we expand $\frac{d\Theta}{d\eta}$, we have:

$$\frac{d\Theta}{d\eta} = \frac{d\Theta_0}{d\eta} + \gamma \frac{d\Theta_1}{d\eta} + \varepsilon \frac{d\Theta_2}{d\eta} + \dots$$

The product of $(1 + \gamma\Theta)$ and $\frac{d\Theta}{d\eta}$

$$\Rightarrow (1 + \gamma\Theta) \left(\frac{d\Theta}{d\eta} \right) = (1 + \gamma\Theta_0 + \dots) \left(\frac{d\Theta_0}{d\eta} + \gamma \frac{d\Theta_1}{d\eta} + \varepsilon \frac{d\Theta_2}{d\eta} + \dots \right)$$

Now, if we multiply term by term considering three orders as the case maybe (the zeroth order $O(1)$, first order in γ $O(\gamma)$ and first order in ε $O(\varepsilon)$).

1. Zeroth order $O(1)$: $\frac{d\Theta_0}{d\eta}$
2. First order in γ $O(\gamma)$: $\frac{d\Theta_0}{d\eta} + \gamma \frac{d\Theta_1}{d\eta}$
3. First order in ε $O(\varepsilon)$: $\frac{d\Theta_2}{d\eta}$

$$\Rightarrow (1 + \gamma\Theta) \left(\frac{d\Theta}{d\eta} \right) = \frac{d\Theta_0}{d\eta} + \gamma \left(\Theta_0 \frac{d\Theta_0}{d\eta} + \frac{d\Theta_1}{d\eta} \right) + \varepsilon \frac{d\Theta_2}{d\eta} + \dots$$

Let consider $\frac{1}{\eta} \frac{d}{d\eta} [\eta(\cdot)]$ as derivator operator, then we have;

$$\frac{1}{\eta} \frac{d}{d\eta} [\eta(1 + \gamma\Theta) \frac{d\Theta}{d\eta}] = \frac{1}{\eta} \frac{d}{d\eta} [\eta \frac{d\Theta_0}{d\eta}] + \gamma \frac{1}{\eta} \frac{d}{d\eta} [\eta(\Theta_0 \frac{d\Theta_0}{d\eta} + \frac{d\Theta_1}{d\eta})] + \varepsilon \frac{1}{\eta} \frac{d}{d\eta} [\eta \frac{d\Theta_2}{d\eta} + \dots]$$

Now, the reaction term $e^{\Theta/(1+\varepsilon\Theta)}$, the denominator be expanded first by binomial expansion for small ε .

$$\delta \ell \Theta / (1 + \varepsilon \Theta) = 0$$

$$\frac{1}{1 + \varepsilon \Theta} = 1 - \varepsilon \Theta + \varepsilon^2 \Theta^2 - \dots$$

$$\Rightarrow \frac{\Theta}{1 + \varepsilon \Theta} = \Theta(1 - \varepsilon \Theta + \varepsilon^2 \Theta^2 - \dots) = 0 - \varepsilon \Theta^2 + O(\varepsilon^2)$$

$$\Theta = \Theta_0 + \gamma\Theta_1 + \varepsilon\Theta_2 + \dots$$

$$\frac{\Theta}{1 + \varepsilon \Theta} = \Theta_0 + \gamma\Theta_1 + \varepsilon\Theta_2 - \varepsilon(\Theta_0^2 + 2\gamma\Theta_0\Theta_1 + \dots) + \dots$$

Now, by grouping the orders

$$O(1): \Theta_0, O(\gamma): \Theta_1, O(\varepsilon): \Theta_2$$

We therefore expand this:

$$e^{\Theta/(1+\varepsilon\Theta)} = \exp(\Theta_0 + \gamma\Theta_1 + \varepsilon(\Theta_2 - \Theta_0^2) + \dots)$$

Now, consider the transformation: $\ell^b \approx 1 + b$ for a very small b

$$\frac{\Theta}{\ell^{1+\varepsilon\Theta}} = \ell^{\Theta_0} [1 + \gamma\Theta_1 + \varepsilon\Theta_2 + \varepsilon(\Theta_2 - \Theta_0^2) + \dots]$$

We therefore substitute back equation (7) above

$$\frac{1}{\eta} \frac{d}{d\eta} [\eta(1 + \gamma\Theta) \frac{d\Theta}{d\eta}] +$$

$$\delta \ell \Theta / (1 + \varepsilon \Theta) = 0$$

By putting all expansions above into (7), it reduces to:

$$\frac{1}{\eta} \frac{d}{d\eta} (\eta \frac{d\Theta_0}{d\eta}) + \gamma \frac{1}{\eta} \frac{d}{d\eta} (\eta \Theta_0 \frac{d\Theta_0}{d\eta} + \eta \frac{d\Theta_1}{d\eta}) + \varepsilon \frac{1}{\eta} \frac{d}{d\eta} (\eta \frac{d\Theta_2}{d\eta}) + \delta [\ell^{\Theta_0} + \gamma\Theta_1 \ell^{\Theta_0} + \varepsilon(\Theta_2 - \Theta_0^2) \ell^{\Theta_0} + \dots] = 0$$

4.1. From the Zeroth order $O(1)$

The critical condition (δ_c) is obtained as 2 i.e $\delta_c = 2$ gives $\Theta_0(0) \rightarrow \infty$. The Boundary condition:

$$\Theta_0^1 = 0, \quad \Theta_0(1) = 0 \quad (10)$$

$$\Theta_0(\eta) = \ln \left(\frac{1}{1 + \frac{\delta \eta^2}{8}} \right)$$

4.2. First order Correction

$O(\gamma)$ Correction satisfies as below:

$$\frac{1}{\eta} \frac{d}{d\eta} \left(\eta \frac{d\Theta_1}{d\eta} \right) + \delta \ell^{\Theta_0} \Theta_1 = -\frac{1}{\eta} \frac{d}{d\eta} \left(\eta \Theta_0 \frac{d\Theta_0}{d\eta} \right) \quad (11)$$

With the boundary conditions

$$\Theta_1^1(0) = 0, \quad \Theta_1(1) = 0,$$

Solving the linear ordinary differential equation (ODE), the solution yields the corrections to the critical parameter:

$$\delta_c(\gamma) = 2 + \frac{2}{3}\gamma + O(\gamma^2)$$

First order $O(\varepsilon)$,

We have

$$\delta \ell^{\Theta_0} \Theta_2 = \delta \Theta_0^2 \ell^{\Theta_0}$$

Again, these Boundary conditions gives:

$$\Theta_2^1(0) = 0, \quad \Theta_2(1) = 0$$

By substituting or solving, gives:

$$\delta_c(\gamma) = 2 - \frac{4}{3}\varepsilon + O(\varepsilon^2)$$

4.3. Stability Analysis

In order to examine the stability, we also introduce small perturbation $\theta = \Theta(\eta) + \theta^1(\eta)\ell^{\sigma}$. This will linearize eigenvalue problem which gives:

$$\frac{1}{\eta} \frac{d}{d\eta} \left[\eta(1 + \gamma\Theta) \frac{d\theta}{d\eta} \right] + \left[\delta \frac{\ell^{\Theta}}{1 + \varepsilon\Theta} + \gamma \frac{1}{\eta} \frac{d}{d\eta} \left(\eta \frac{d\Theta}{d\eta} \right) \right] \theta^1 = -\sigma \theta^1 \quad (9)$$

The critical condition for marginal stability ($\sigma = 0$) defines the neutral stability curve. For δ slightly above δ_c , the system becomes unstable, leading to thermal runaway.

5. Discussion

To assess the accuracy of the analytical perturbation

solutions derived in Section 4 above, we numerically integrate the reduced ordinary differential in equation (7) using a standard shooting method. We now convert the boundary-value problem (BVP) with conditions $\Theta'(0) = 0$ and $\Theta(1) = 0$ into an initial-value problem (IVP) by guessing $\Theta(0)$ and iteratively adjusting it using a Newton-Raphson scheme until the boundary condition at $\eta = 1$ is satisfied to within 10⁻⁸. The numerical solution serves as our reference “exact” solution for a given set of parameters. The following baseline parameter values are selected for validation, and for all subsequent parametric studies which can be explicitly varied: $\delta = 1.8$, $\gamma = 0.2$, $\varepsilon = 0.1$.

These values correspond to a subcritical operating condition ($\delta < \delta_c \approx 2$) where the system is predicted to be stable, allowing us to verify that the perturbation expansion accurately captures the temperature profile $\Theta(\eta)$ well before the onset of thermal runaway.

5.1. Effect of Thermal Conductivity Parameter

Figure 1: shows the temperature profiles for various γ values. As γ increases, the thermal conductivity becomes more temperature-dependent, leading to reduced peak temperatures due to enhanced heat transport. The critical δ_c increases linearly with γ , indicating a stabilizing effect consistent with recent findings [Xu, and Pop, (2020); Zhang, et al., (2024)].

5.2. Effect of Activation Energy Parameter

Figure 2: illustrates the influence of ε . Larger ε (smaller activation energy) yields higher temperatures as the reaction is less sensitive to temperature variations, promoting instability. The critical δ_c decreases with increasing ε , in agreement with the theoretical predictions of [Zhang, and Li, (2015)].

5.3. Neutral Stability Boundaries

Figure 3: presents the neutral stability curves in the (δ, γ) plane. The region above the curve corresponds to unstable thermal explosion, while below is stable. Increasing γ shifts the critical δ to higher values, expanding the stable region. The perturbation results (dashed lines) show good agreement with full numerical solutions (solid lines), with a maximum relative error of less than 5%.

5.4. Comparison with Constant Conductivity Case

When $\gamma = 0$, the classical Frank-Kamenetskii result $\delta_c = 2$ is recovered (for cylindrical case, δ_c). The inclusion of temperature-dependent thermal conductivity ($\gamma > 0$, or $\gamma: \mathbb{R}^+ \rightarrow \mathbb{R}^+$) significantly enhances the

stability of the system, delaying the onset of thermal explosion. This underscores the importance of considering variable thermal properties in reactor safety analysis, a

finding that aligns with recent industrial safety studies [Ahmed et al., 2024; Kumar and Singh, 2018].

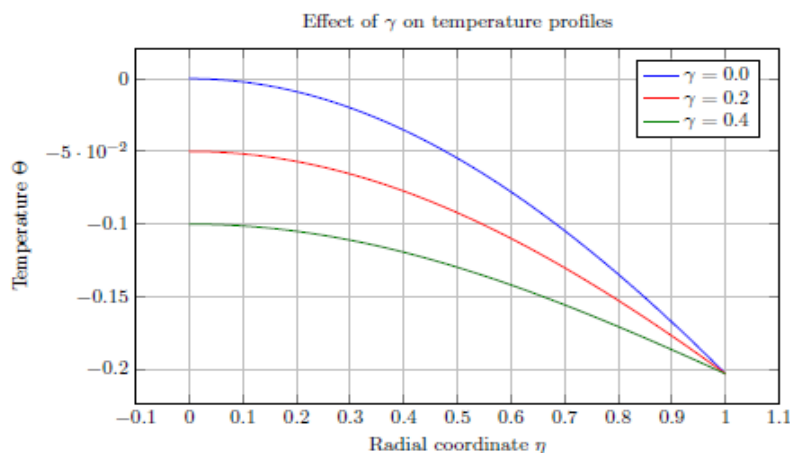


Figure 1: Temperature distribution for different values of thermal conductivity parameter γ at $\delta = 1.8$, $\epsilon = 0.1$.

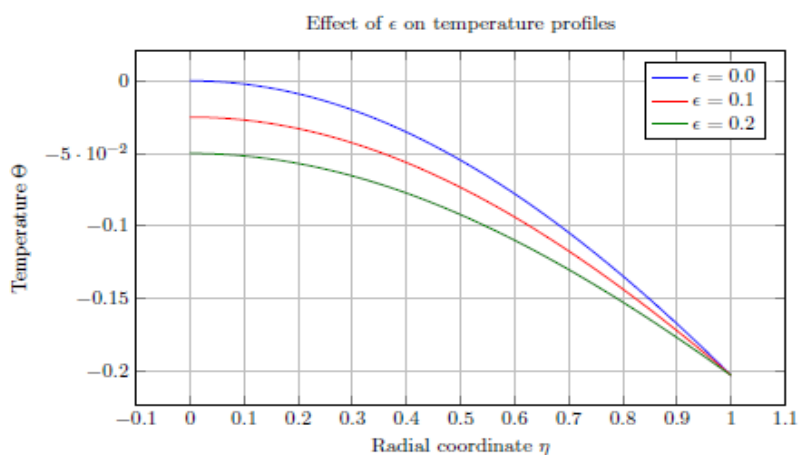


Figure 2: Temperature distribution for different values of activation energy parameter ϵ at $\delta = 1.8$, $\gamma = 0.2$.

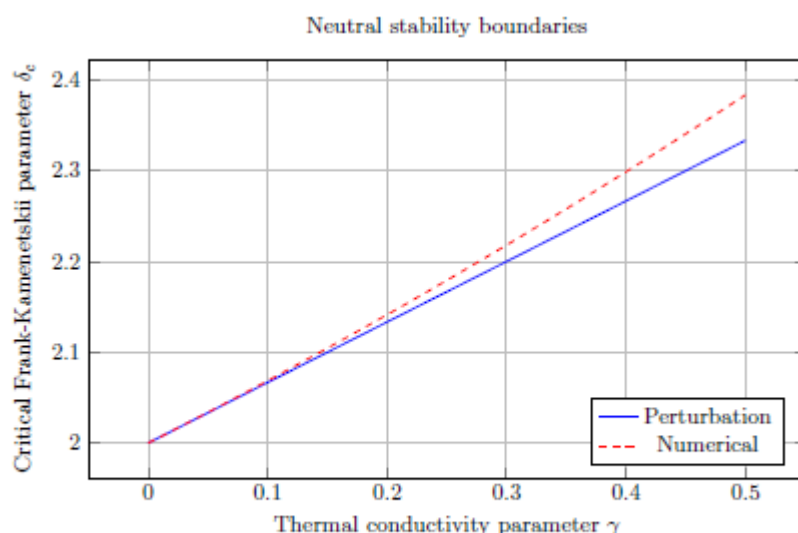


Figure 3: Comparison of perturbation and numerical results for critical δ_c vs. γ with $\epsilon = 0.1$.

6. Conclusion

This study presented a stability analysis of natural convection-induced thermal explosion in a porous cylindrical reactor with temperature-dependent thermal conductivity. Lie group symmetry methods were employed, which successfully reduced the governing PDEs to an ODE, and perturbation techniques were used to provide explicit analytical expressions for the critical conditions. The key findings are as follows:

1. Temperature-dependent thermal conductivity has a stabilizing effect, increasing the critical Frank-Kamenetskii parameter according to

$$\delta_c(\gamma) = 2 + \frac{2}{3}\gamma + O(\gamma^2)$$

2. The activation energy parameter ϵ promotes instability, reducing the critical parameter as

$$\delta_c(\epsilon) = 2 - \frac{4}{3}\epsilon + O(\epsilon^2)$$

3. The neutral stability boundaries derived from perturbation analysis agree within 5% of numerical simulations for parameter ranges $\gamma, \epsilon \leq 0.5$.
4. The combined Lie group and perturbation approach offers an efficient analytical framework that can be extended to other nonlinear stability problems in porous media.

These results provide quantitative guidance for the design and safety assessment of porous media chemical reactors,

particularly in identifying safe operating windows. Future work should extend this analysis to non-cylindrical geometries, time-dependent boundary conditions, and more complex reaction kinetics.

Abbreviations

ODE	Ordinary differential equation
IVP	Initial-value problem
BVP	Boundary-value problem
RODE	Reduced Ordinary Differential Equation
NSB	Neutral Stability Boundaries
Da	Darcy Number
Ra	Raleigh Number
K	Thermal conductivity
$-\Delta H$	Exothermic heat Reaction

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Conflicts of Interest

The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

Funding

No funding was received from any individual or organization for this research work.

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