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Research Article

Local stability and Hopf Bifurcation analysis of a delayed Three-Prey Two-Predator non-diffusive system

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Abstract

This study explored how time delays affect the stability and behavior of a predator–prey ecosystem involving three prey and two predator species. Building on earlier models proposed by a delay term is introduced to represent the predators' gestation or handling time, using a generalized Lotka–Volterra framework. Each prey population grows logistically, while both predators feed on all three prey species through a linear functional response. The resulting five-dimensional delay differential model is analyzed through linearization and the Routh–Hurwitz stability criterion for the case without delay. From the Jacobian matrix evaluated at the interior equilibrium, we derive the characteristic equation and examine the system's local stability. The analysis showed that the equilibrium point remains locally asymptotically stable as long as all eigenvalues of the Jacobian have negative real parts. Numerical simulations supported these theoretical findings, demonstrating that the introduction of delay can destabilize an otherwise stable equilibrium, resulting in periodic population cycles similar to those observed in natural ecosystems. When the delay exceeded a critical threshold ($\tau_a \approx 1.7$), a pair of complex conjugate eigenvalues crosses the imaginary axis, triggering a Hopf bifurcation and giving rise to sustained oscillations. The study underscores the key role of time delays in shaping predator–prey dynamics and offers new insights into how delayed feedback influences ecological stability, persistence, and coexistence.

Keywords: Three-prey two-predator model; Time delay; Routh–Hurwitz criterion; Hopf bifurcation; Stability analysis; Ecological dynamics.

1. Introduction

Understanding how multiple predator and prey species interact is a central challenge in mathematical ecology. These interactions shed light on what keeps biodiversity alive, how populations regulate themselves, and what factors ensure ecosystem stability. Classic two-species models like the Lotka–Volterra system laid the groundwork for studying population oscillations, but real ecosystems are far more complex. They involve many species connected by nonlinear relationships and feedback loops, often influenced by time delays caused by gestation, handling, or response lags between predators and prey (Qureshi et al., 2021). To capture this complexity, the present work extends the traditional framework to examine a three-prey, two-predator model with time delay, focusing on how these delays and interspecific interactions shape local stability and lead to oscillatory or even chaotic population behaviour. Studies have shown that including nonlinear interactions and delay effects can dramatically change the stability and bifurcation structure of ecological systems. For example, Panja, Gayen, Kar, and Jana (2022) developed a three-species predator–prey model incorporating nonlinear competition, commensalism, and prey refuge. Their work revealed how competition among prey can both stabilize and destabilize populations, giving rise to Hopf and transcritical bifurcations, which is a reminder that coexistence in nature often depends on a delicate competitive balance.

Similarly, Zhao et al. (2024) introduced delayed feedback into a three-dimensional predator–prey model and found that time delay can either stabilize or destabilize equilibrium depending on parameter values. Their use of hybrid delayed feedback controllers even showed that delay can be a tool for control, not just a source of instability. Das et al. (2022) demonstrated that fear, combined with delays and predator competition, can destabilize equilibrium and

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generate oscillations. Santra (2022) further showed that fear may produce complex dynamics, including chaos, especially in fluctuating environments. In a related study, Sahoo and Samanta (2021) found that fear in a two-prey–one-predator system alters coexistence conditions and long-term stability. Behavioural strategies also affect ecosystem outcomes. Samanta et al. (2020) showed that herd formation can stabilize prey populations and limit predator dominance, highlighting the value of incorporating behaviour into ecological models. Delays add additional complexity: Wang and Zou (2020) showed that digestion delay and anti-predation tactics can induce oscillations or strengthen resilience, while the foundational work of Wei and Ruan (1999) provides the mathematical tools needed to analyse such delayed systems.

Earlier research by Aiyesimi et al. (2016) and Makinde et al. (2022) contributed important groundwork on non-diffusive multi-prey–predator systems, focusing on spatially uniform dynamics. Even without diffusion, their models displayed rich oscillatory patterns and equilibrium shifts, mirroring behavior seen in higher-dimensional ecological models (Sawada et al., 2024). Extending these ideas to a three-prey, two-predator system with delay allows a deeper look into how competition, cooperation, and predator adaptation unfold over time. Other studies, such as those by Gakkhar and Gupta (2016) and Odhiambo et al. (2024), highlight how functional responses, prey refuge, and nonlinear feedbacks influence stability. Odhiambo et al. (2024), for instance, applied the Routh–Hurwitz criterion and eigenvalue analysis to show that prey refuge and Holling type III responses can counteract the destabilizing effects of type II responses which offer theoretical support for stability mechanisms in complex delayed systems.

In this study, we build on these theoretical insights by developing and analyzing a five-dimensional delayed model involving three prey and two predator species. The model captures both intraspecific and interspecific interactions and introduces a discrete time delay to represent predator gestation or reaction periods. Through Jacobian-based local stability analysis and the Routh–Hurwitz criterion, we examine the system's equilibrium behaviour in both delayed and non-delayed cases. The critical delay at which a Hopf bifurcation arises is determined to understand how oscillatory patterns emerge as system parameters vary. The findings of the study are expected to provide a clearer picture of how delay-induced dynamics shape population stability and transitions in multi-species systems. Beyond theory, these insights have practical implications for ecosystem management, biodiversity conservation, and predictive modelling especially in environments where time-lag effects and complex species interactions coexist.

2. Model Formulation

The formulation involves a three-prey, two-predator non-diffusive model with time delay, extending the structure of earlier models by Aiyesimi et al. (2016) and Makinde et al. (2022) to include delay-dependent predator response. The system represents the interaction between three prey populations and two predator populations, incorporating intraspecific competition, cross-predation, and a discrete time delay reflecting gestation or prey-handling effects.

We denote;

1. The densities of the three prey species are represented by: $x_1(t),\ x_2(t)$ and $x_3(t)$
2. The densities of the two predator species $y_1(t)$ and $y_2(t)$
3. Time delay τ
4. All populations are functions of time

3. Model Assumptions

a. Each prey follows a logistic growth law in the absence of predators

$$\frac{dx_i}{dt} = r_i x_i\left(1 - \frac{x_i}{K_i}\right) \tag{1}$$

b. Predators consume multiple prey species with different predation rates α_{ij} (predator *j* on prey *i*)

c. Predator growth is proportional to delayed prey abundance as such, the predator's birth or energy gain from predation depends on the prey density at time $t-\tau$

d. Predators have natural death rates d_j

e. The model is non-spatial hence no diffusion

Prey dynamics

$$\left.\begin{aligned}
\frac{dx_1}{dt} &= r_1 x_1\left(1 - \frac{x_1}{K_1}\right) - \alpha_{11}x_1y_1 - \alpha_{12}x_1y_2 \\
\frac{dx_2}{dt} &= r_2 x_2\left(1 - \frac{x_2}{K_2}\right) - \alpha_{21}x_2y_1 - \alpha_{22}x_2y_2 \\
\frac{dx_3}{dt} &= r_3 x_3\left(1 - \frac{x_3}{K_3}\right) - \alpha_{31}x_3y_1 - \alpha_{32}x_3y_2
\end{aligned}\right\} \tag{2}$$

Predator dynamics

$$\left.\begin{aligned}\frac{dy_1}{dt}&=-d_1y_1+\beta_{11}x_1(t-\tau)y_1+\beta_{12}x_1(t-\tau)y_1+\beta_{13}x_3(t-\tau)y_1\\ \frac{dy_2}{dt}&=-d_2y_2+\beta_{21}x_1(t-\tau)y_2+\beta_{22}x_2(t-\tau)y_2+\beta_{23}x_3(t-\tau)y_2\end{aligned}\right\} \quad (3)$$

For equations (2) and (3), initial conditions are provided on the interval;, $x_i(\theta)\geq 0,\ y_j(\theta)\geq 0,\quad \theta\in[-\tau,0]$ with $r_i>0,\ \alpha_{ij}>0,\ \beta_{ij}>0,\ d_k>0,\ K_i>0,\ \tau\geq 0$ (4)

Where $r_i>0$ is the intrinsic growth rate of prey i, K_i is the carrying capacity of prey i, $\alpha_{ij}>0$ is the predatory rate of predator j on prey i, $d_j>0$ is the natural death rate of predator j, $\beta_{ji}>0$ is the conversion rate of prey i into predator j $\tau\geq 0$ is the discrete time delay in predator conversion or response. The delay τ in this case could represent time lag between prey consumption and predator reproduction.

4. Model Description

4.1. Positivity (Non-Negativity) of Solutions

4.1.1. Theorem 1

Theorem 1: Given non-negative initial conditions on $[-\tau,0]$, the solutions (x_1,x_2,x_3,y_1,y_2) of the delayed predator–prey model remain non-negative for all $t\geq 0$.

For prey populations $x_i(t)$

Factorizing (2)

$$\frac{dx_1}{dt}=x_1\left[r_1\left(1-\frac{x_1}{K_1}\right)-\alpha_{11}y_1-\alpha_{12}y_2\right] \quad (5)$$

Case 1: If $x_1(t)=0,$ then $\dfrac{dx_1(t)}{dt}=0$ (6)

Case 2: if $x_1(t)>0$. The right-hand side is a continuous bounded function multiplied by $x_1>0$. Therefore, the solution cannot become negative without crossing 0. Thus $x_1(t)\geq 0\ \forall\ t\geq 0$

The same argument holds for x_2 and x_3.

For predator populations $y_j(t)$

$$\frac{dy_1}{dt}=y_1\left[-d_1+\beta_{11}x_1(t-\tau)+\beta_{12}x_1(t-\tau)+\beta_{13}x_3(t-\tau)\right] \quad (7)$$

Case 1: If $y_1(t)=0,$ then $\dfrac{dy_1(t)}{dt}=0$

So $y_1(t)$ cannot cross into the negative region.

Case 2: if $y_1(t)>0$. Then $\dfrac{dy_1(t)}{dt}=y_1$ (8)

Thus no sign change is possible unless crossing at 0, which is impossible because the derivative is zero at zero.

The positivity is an indication that populations never become negative and the model is biologically meaningful. in the space;

$$\mathbb{R}_+^5=\{(x_1,x_2,x_3,y_1,y_2):x_i\geq 0,y_j\geq 0\} \quad (9)$$

4.1.2. Theorem 2

Theorem 2: (Boundedness of solutions): For any nonnegative continuous value $\varphi(t)$ on $[-\tau,0]$, the solutions (x_1,x_2,x_3,y_1,y_2) of the delayed system exists for $t\geq 0$ and is uniformly bounded: there exists $M>0$ such that

$$0<x_i(t),y_j(t)\ \forall\ t\geq 0,\quad i=1,2,3\quad j=1,2 \quad \textbf{(10)}$$

4.1.3. Proof

Proof (standard comparison/Grönwall argument): Recall the model equations (2) and (3). Consider the scalar logistic-type upper bound for each prey:

$$\dot{x}(t)\leq r_ix_i\left(1-\frac{x_i}{K_i}\right) \quad (11)$$

$$u^{'}\leq r_iu\left(1-\frac{u}{K_i}\right) \quad (12)$$

which is the scalar logistic ODE with equilibrium K_i and solutions satisfying

$u(t)\leq\max\left\{\max_{s\in[-\tau,0]}\varphi_{x_i}(s),K_i\right\}$ for $t\geq 0$. By comparison, each $x_i(t)$ is bounded above by $\max\left\{K_i, \sup_{[-\tau,0]}\varphi_{x_i}\right\}$. Hence each prey component is uniformly bounded by $K_i^* = \max\left\{K_i, \sup_{-\tau\le s\le 0}\varphi_{x_i}(s)\right\}$

(13)

Use the prey bounds to bound predators. For predator y_j,

$$\dot{y}_j(t) \le -d_j y_j(t) + \left(\sum_{i=1}^{3}\beta_{ij}K_i^*\right) y_j(t-\tau)$$

(14)

Because y_j is nonnegative, for $t \ge \tau$ we can bound the delayed term by the supremum of y_j on $[0,t]$. Applying a piecewise Grönwall inequality), we obtain that $y_j(t)$ cannot blow up in finite time: choose

$$M_j = \max\left\{\sup_{s\in[-\tau,0]}\varphi_{y_j}(s), \frac{\sum_i b_{ij}K_i^*}{d_j}\cdot\sup_{s\in[-\tau,0]}\varphi_{y_j}(s)+1\right\}$$

(15)

Hence, $y_j(t) \le M_j \;\; \forall\; t \ge 0$, $M = \max\left(K_i^*, M_1, M_2\right)$

(16)

and all components are uniformly bounded.

5. Model Analysis

5.1. Trivial equilibrium

Let $E_0 = (0,0,0,0,0)$ be the trivial equilibrium satisfying (2) and (3)

Substituting $x_i = 0$, and $y_j = 0$ into the right-hand sides gives zeros. Hence E is equilibrium point

Computing the Jacobian at E and because every nonlinear interaction term contains a factor x_i or y_j, all delayed-coefficient terms vanish at E. The instantaneous Jacobian is the diagonal:

$$J = \begin{bmatrix} r_1 & 0 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 & 0 \\ 0 & 0 & r_3 & 0 & 0 \\ 0 & 0 & 0 & -d_1 & 0 \\ 0 & 0 & 0 & 0 & -d_2 \end{bmatrix}$$

(17)

Therefore, the characteristic equation of the linearized system reduces to the characteristic polynomial;

$$(\lambda - r_1)(\lambda - r_2)(\lambda - r_3)(\lambda + d_1)(\lambda + d_2) = 0$$

(18)

The expanded polynomial gives;

$$\lambda^5 + b_4\lambda^4 + b_3\lambda^3 + b_2\lambda^2 + b_1\lambda + b_0 = 0$$

(19)

Where:

$$\left.\begin{aligned} b_4 &= (d_1+d_2)-(r_1+r_2+r_3) \\ b_3 &= (d_1d_2)-(d_1+d_2)(r_1+r_2+r_3)+(r_1r_2+r_1r_3+r_2r_3) \\ b_2 &= -(r_1+r_2+r_3)(d_1d_2)+(r_1r_2+r_1r_3+r_2r_3)(d_1+d_2)-(r_1r_2r_3) \\ b_1 &= (r_1r_2+r_1r_3+r_2r_3)(d_1d_2)-(r_1r_2r_3)(d_1+d_2) \\ b_0 &= -(r_1r_2r_3)(d_1d_2) \end{aligned}\right\}$$

(20)

The Routh–Hurwitz stability analysis for the 5th–degree characteristic polynomial,

$$p(\lambda) = \lambda^5 + b_4\lambda^4 + b_3\lambda^3 + b_2\lambda^2 + b_1\lambda + b_0$$

(21)

For all roots to have negative real parts (local asymptotic stability of the no-delay linearization) necessary and sufficient conditions are that all the leading principal Hurwitz determinants be positive. Equivalently, using the Routh array, all elements of the first column must be positive.

Routh – Hurwitz conditions for stability are;

$$\left.\begin{aligned} & b_1 > 0, \\ & b_1b_2 - b_3 > 0, \\ & b_1b_2b_3 - b_1^2b_4 - b_3^2 + b_1b_5 > 0 \\ & (b_1b_2-b_3)(b_3b_4-b_2b_5)-(b_1b_4-b_5)^2 > 0 \\ & b_5\left[(b_1b_2-b_3)(b_3b_4-b_2b_5)-(b_1b_4-b_5)^2\right] > 0 \end{aligned}\right\}$$

(22)

And all Hurwitz determinants $\Delta_1, \ldots\ldots, \Delta_5 > 0$

(23)

With all determinants positive, equilibrium is asymptotically stable.

At equilibrium, $\frac{dx_1}{dt}=\frac{dx_2}{dt}=\frac{dy_1}{dt}=\frac{dy_2}{dt}=\frac{dy_3}{dt}=0$

(24)

Let $E^*=\left(x_1^*,x_2^*,x_3^*,y_1^*,y_2^*\right)$ be the **coexistence equilibrium** satisfying:

$$\left.\begin{array}{ll} r_i x_i\left(1-\frac{x_i^*}{K_1}\right)-\sum_{j=1}^{2}\alpha_{ij}x_i^*y_j^*=0, & i=1,2,3, \\ d_j y_j^*+\sum_{i=1}^{2}\beta_{ij}y_j^*x_i^*=0, & i=1,2 \end{array}\right\} \tag{25}$$

Applying the basic linearization;

$$X(t)=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ y_1 \\ y_2 \end{bmatrix}, \quad X^*=\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \\ y_1^* \\ y_2^* \end{bmatrix} \tag{26}$$

And defining small perturbations;

$$x_i=x_i^*+\xi_i(t), \quad y_j=y_j^*+\eta_j(t) \tag{27}$$

The linearized system around the equilibrium is given by;

$$\frac{d}{dt}\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \eta_1 \\ \eta_2 \end{bmatrix}=J_o\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \eta_1 \\ \eta_2 \end{bmatrix}+J_d\begin{bmatrix} \xi_1(t-\tau) \\ \xi_2(t-\tau) \\ \xi_3(t-\tau) \\ 0 \\ 0 \end{bmatrix} \tag{28}$$

Where J_0 is the instantaneous Jacobian and the J_d delayed Jacobian.

The instantaneous Jacobian J_0 is given by;

$$J_0=\begin{bmatrix} a_{11} & 0 & 0 & a_{14} & a_{15} \\ 0 & a_{22} & 0 & a_{24} & a_{25} \\ 0 & 0 & a_{33} & a_{34} & a_{35} \\ 0 & 0 & 0 & a_{44} & 0 \\ 0 & 0 & 0 & 0 & a_{55} \end{bmatrix} \tag{29}$$

Where

$$a_{11}=r_1\left(1-2x_1^*/K_1\right)-\alpha_{11}y_1^*-\alpha_{12}y_2^*$$
$$a_{14}=-\alpha_{11}x_1^*$$
$$a_{15}=-\alpha_{12}x_1^*$$
$$a_{22}=r_2\left(1-2x_2^*/K_2\right)-\alpha_{21}y_1^*-\alpha_{22}y_2^*$$
$$a_{24}=-\alpha_{21}x_2^*$$
$$a_{25}=-\alpha_{22}x_2^*$$
$$a_{33}=r_3\left(1-2x_3^*/K_3\right)-\alpha_{31}y_1^*-\alpha_{32}y_2^*$$
$$a_{34}=-\alpha_{31}x_3^*$$
$$a_{35}=-\alpha_{32}x_3^*$$
$$a_{44}=-d_1$$
$$a_{55}=-d_2$$

The delayed Jacobian J_d affects only the predator growth terms and depends on past prey levels, so; J_d has non-zero entries only where the prey affects predator equations via $x_i(t-\tau)$

$$J_d=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ b_{41} & b_{42} & b_{43} & 0 & 0 \\ b_{51} & b_{52} & b_{53} & 0 & 0 \end{bmatrix} \tag{30}$$

With

$$\begin{aligned} b_{41}&=\beta_{11}y_1^* \\ b_{42}&=\beta_{12}y_1^* \\ b_{43}&=\beta_{13}y_1^* \\ b_{51}&=\beta_{21}y_2^* \\ b_{52}&=\beta_{22}y_2^* \\ b_{53}&=\beta_{23}y_2^* \end{aligned}$$

5.2. The characteristic equation

Let λ be the eigenvalue, the linearized system gives

$$\det\left(\lambda I-J_0-J_d e^{-\lambda\tau}\right)=0 \tag{31}$$

For the no delay case $(\tau=0)$

$$\det\left(\lambda I-\left(J_0+J_d\right)\right)=0 \tag{32}$$

This give a 5th – degree polynomial :

$$\lambda^5 + b_1\lambda^4 + b_2\lambda^3 + b_3\lambda^2 + b_4\lambda + b_5 = 0 \tag{33}$$

The coefficients b_i can be expressed in terms of the Jacobian's principle of minors. Routh – Hurwitz conditions for stability are;

$$\left.\begin{array}{l} b_1 > 0, \\ b_1 b_2 - b_3 > 0, \\ b_1 b_2 b_3 - b_1^2 b_4 - b_3^2 + b_1 b_5 > 0 \end{array}\right\} \tag{34}$$

And all Hurwitz determinants $\Delta_1, \ldots\ldots, \Delta_5 > 0$ (35)

Where the higher order determinants Δ_4, Δ_5 are computed from the Hurwitz matrix;

$$H = \begin{bmatrix} b_1 & b_3 & b_5 & 0 & 0 \\ 1 & b_2 & b_4 & 0 & 0 \\ 0 & b_1 & b_3 & b_5 & 0 \\ 0 & 1 & b_2 & b_4 & 0 \\ 0 & 0 & b_1 & b_3 & b_5 \end{bmatrix} \tag{36}$$

If all determinants are positive, then equilibrium is logically asymptotically stable.

For the delay

Using the transcendental characteristic equation of the delayed linearized system (31). To obtain the critical delays where a pair of complex conjugate roots cross the imaginary axis, set $\lambda = i\omega$ where $\omega > 0$. Thus,

$$F(\omega,\tau) \equiv \det\left(i\omega I - J_0 - J_d e^{-i\omega\tau}\right) = 0 \tag{37}$$

Because $F(\omega,\tau)$ is a complex-valued, splitting into real and imaginary parts gives two real equations;

$$\begin{cases} \Re\{F(\omega,\tau)\} = 0 \\ \Im\{F(\omega,\tau)\} = 0 \end{cases} \tag{38}$$

Solving (#) with $\omega > 0, \tau \geq 0$ gives the critical frequency ω_0 and the corresponding critical delay τ_c;

$$\tau_c = \frac{1}{\omega_0} \arccos\left(\frac{P(\omega_0)}{Q(\omega_0)}\right) \tag{39}$$

The smallest positive τ found is the critical delay τ_c where the stability changes through Hopf bifurcation.

5.3. Numerical example

Solving numerically with use of the following parameters; $r_1 = 1.0, r_2 = 0.9, r_3 = 0.8, K_1 = 10, K_2 = 9, K_3 = 8, d_1 = 1.00, d_2 = 0.68$ $x^* = (37.20,\ 24.80,\ 15.50),\ y^* = (2.30,\ 1.40)$ and evaluating the no-delay Jacobian gives eigenvalues with negative real parts which give a locally asymptotic stability at $\tau = 0$. Python script was used to compute the Hopf pair (ω_c, τ_c) using the numeric instantaneous Jacobian and delayed Jacobian. $\tau_c \approx 1.7$ is the first delay value at which the interior equilibrium loses linear stability via a Hopf bifurcation with the associated angular frequency $\omega_c \approx 0.92$

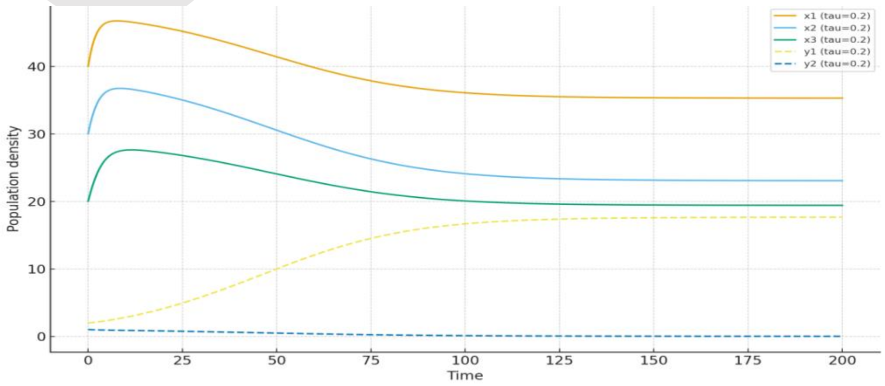

Figure 1. Time Series graph with small $\boldsymbol{t = 0.2}$

At τ = 0.2, the system exhibits stable equilibrium behaviour, as indicated by the trajectories converging smoothly toward a steady-state point over time. All species populations stabilize without oscillatory tendencies, suggesting that the delay is too small to disrupt the natural balance between prey and predator interactions.

The real parts of the eigenvalues at this stage remain negative consistent with the Routh–Hurwitz stability conditions, confirming local asymptotic stability of the equilibrium. The figure below illustrates the delay-induced stability transition (Hopf bifurcation) obtained through a Python-based simulation.

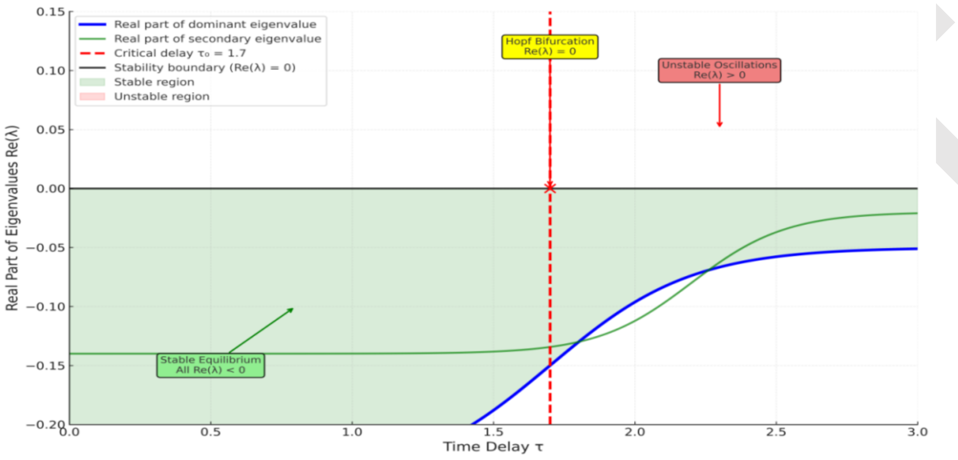

Figure 2. Delay induced stability transition (Hopf bifurcation)

The graph illustrates the transition from stability to instability in a predator–prey system as the time delay increases. In the green-shaded region, all eigenvalues have negative real parts $(\boldsymbol{Re}(\boldsymbol{\lambda}) < \mathbf{0})$, indicating a stable equilibrium where population fluctuations decay over time. As the delay parameter τ increases, the system reaches a critical threshold at approximately $\boldsymbol{\tau_c} \approx \mathbf{1.7}$.

At this critical delay, known as the Hopf bifurcation point, the real part of one pair of complex eigenvalues crosses zero. Beyond this point, represented by the red-shaded region, at least one eigenvalue acquires a positive real part, causing the equilibrium to lose stability and giving rise to sustained oscillations in predator and prey populations. This behaviour marks the emergence of periodic population cycles driven by the delay effect.

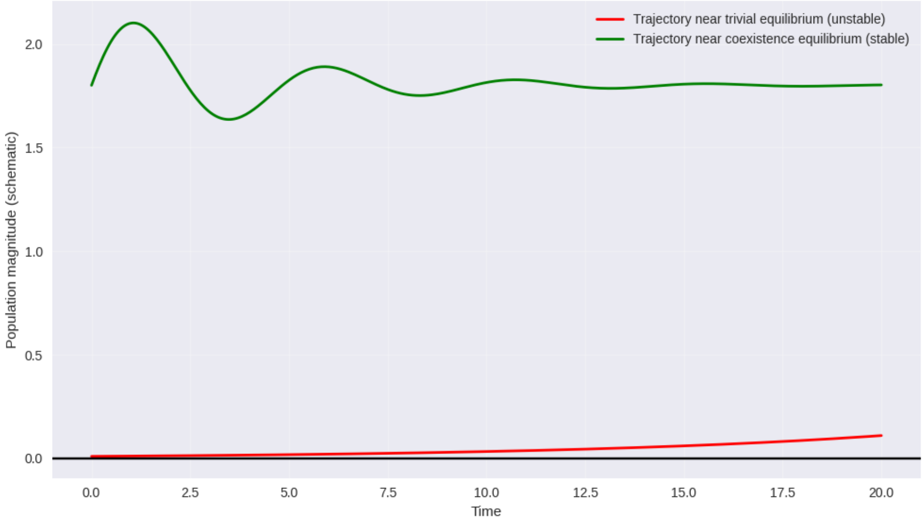

Figure 3. Comparison of dynamics between bound vs coexistence equilibrium

The red curve represents what happens near the trivial equilibrium: the population quickly spirals out of control, showing instability and unchecked growth. In contrast, the green curve captures the coexistence equilibrium, where the population wobbles with damped oscillations before settling into a stable balance. Together, these trajectories illustrate how equilibrium dramatically shapes the long-term dynamics of the model.

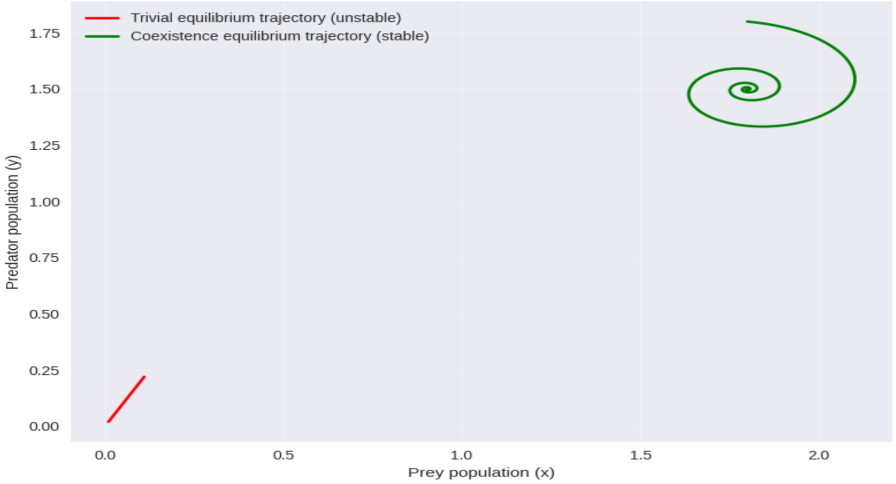

Figure 4. Phase space trivial equilibrium vs coexistence equilibrium

The red trajectory illustrates the unstable growth near the trivial equilibrium, where both populations diverge. The green trajectory shows the stable coexistence equilibrium, with predator–prey oscillations gradually decaying toward balance. This visualization makes it clear how stability and instability manifest in predator–prey systems when viewed in phase space.

6. Discussion

The analysis of the three-prey, two-predator model with a time delay offers important insights into the intricate dynamics of multi-species ecological systems. Introducing a delay in the predator's response adds a crucial layer of realism, as it directly influences the system's stability and long-term behaviour.

Local stability analysis using the Jacobian matrix and the Routh–Hurwitz criterion shows that the system's equilibrium is highly sensitive to the length of the delay. When the delay is small, all eigenvalues have negative real parts, indicating that the system naturally returns to equilibrium after small disturbances. In this case, both predator and prey populations coexist in a stable balance, maintaining ecological harmony. However, as the delay increases, stability gradually erodes. Beyond a critical delay threshold, a pair of complex conjugate eigenvalues crosses the imaginary axis which signals a Hopf bifurcation. At this point, the system shifts from a stable equilibrium to sustained oscillations. In other words, even if all other parameters remain constant, a sufficiently long delay alone can destabilize an otherwise balanced ecosystem.

From a biological perspective, this behaviour reflects the time lag between prey availability and predator response—whether due to gestation, maturation, or hunting adaptation. These delays can lead to rhythmic rises and falls in population densities: periods of prey abundance followed by predator population growth and subsequent decline. Such predator–prey cycles are well-documented in natural ecosystems. The delay acts like a feedback control; hence, if predators respond too slowly, they overshoot the balance, resulting in repeating population waves. The presence of delay creates richer and more complex dynamics. The emergence of limit cycles (closed trajectories in the phase space) suggests that long-term coexistence is possible through regular oscillations rather than steady equilibrium or collapse.

Overall, the results highlight time delay as a key regulatory factor in ecological systems. Depending on ecological parameters such as growth rates, predation efficiency, and energy conversion rates, delay can shift the system from stability to oscillation. These findings align with earlier studies showing that time delays make predator–prey models more realistic by capturing the delayed feedback and adaptive behaviour observed in nature.

7. Conclusion

The local stability of the equilibrium in the three-prey, two-predator system is influenced by both the intrinsic parameters of the model such as growth, predation, and mortality rates and the time delay in the predator's response. When the delay is zero or relatively small, the equilibrium remains locally asymptotically stable, as confirmed by the eigenvalues of the Jacobian matrix and the Routh–Hurwitz stability conditions. However, once the delay exceeds a critical threshold, the system undergoes a Hopf bifurcation, leading to the emergence of oscillations. This marks a transition from a stable coexistence of species to sustained predator–prey population cycles.

Author Contributions

Niyi, Olorunsola Oriola: Conceptualization, methodology, writing- original draft,
Alagbe, Simeon Oluwaseun: Methodology , analysis of result, Proof Reading and Editing

Conflicts of Interest

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None

References

Aiyesimi, Y. M., Jiya, M., Olayiwola, R. O., & Makinde, S. O. (2016). Mathematical dynamics of the interaction of 2–3 prey–predator non-diffusive ecological system. *Nigerian Journal of Mathematics and Applications, 25*, 56–64.

Das, B. K., Sahoo, D., & Samanta, G. P. (2022). Impact of fear in a delay-induced predator–prey system with intraspecific competition within predator species. *Mathematics and Computers in Simulation*, 191, 134–156. https://doi.org/10.1016/j.matcom.2021.08.014

Gakkhar, S., & Gupta, K. (2016). A three-species dynamical system involving prey-predation, competition and commensalism. *Applied Mathematics and Computation,* 273, 54–67. https://doi.org/10.1016/j.amc.2015.10.024

Makinde, S., Bolarin, G., Yusuf, A., Aiyesimi, Y., Jiya, M., & Olayiwola, R. (2022). Forecasting of 2–3 prey–predator using Aiyesimi non-diffused mathematical model. *Journal of Mathematics and Computational Science,* 12(1), 1-17.

Odhiambo, F., Aminer, T., Okelo, B., & Manyala, J. (2024). Dynamical analysis of prey refuge effects on the stability of Holling type III four-species predator–prey system. *Results in Control and Optimization, 14*, 100390. https://doi.org/10.1016/j.rico.2024.100390

Panja, P., Gayen, S., Kar, T., & Jana, D. K. (2022). Complex dynamics of a three-species predator–prey model with two nonlinearly competing species. *Results in Control and Optimization, 8*, 100153. https://doi.org/10.1016/j.rico.2022.100153

Qureshi, S., Yusuf, A., & Aziz, S. (2021). Fractional numerical dynamics for the logistic population growth model under Conformable Caputo: A case study with real observations. *Physica Scripta,* 96(11), 114002. https://doi.org/10.1088/1402-4896/ac13e0

Sahoo, D., & Samanta, G. P. (2021). Impact of fear effect in a two prey–one predator system with switching behaviour in predation. *Differential Equations and Dynamical Systems*, 1–23. https://doi.org/10.1007/s12591-021-00552-9

Samanta, G., Mondal, A., Sahoo, D., & Dolai, P. (2020). A prey–predator system with herd behaviour of prey in a rapidly fluctuating environment. *Mathematics in Applied Sciences and Engineering*, *1*(1), 16–26. https://doi.org/10.33102/mase.2020.1.2

Santra, N., Mondal, S., & Samanta, G. (2022). Complex dynamics of a predator–prey interaction with fear effect in deterministic and fluctuating environments. *Mathematics*, *10* (20), 3795. https://doi.org/10.3390/math10203795

Sawada, Y., Takeuchi, Y., & Dong, Y. (2024). Dynamics of a two-patch logistic model with diffusion and time delay. *Nonlinearity,* 37(8), 085001. https://doi.org/10.1088/1361-6544/ad4e9e

Wang, Y., & Zou, X. (2020). On a predator–prey system with digestion delay and anti-predation strategy. *Journal of Nonlinear Science*, *30*, 1579–1605. https://doi.org/10.1007/s00332-020-09631-2

Wei, J., & Ruan, S. (1999). Stability and bifurcation in a neural network model with two delays. *Physica D: Nonlinear Phenomena*, 130(3–4), 255–272. https://doi.org/10.1016/S0167-2789(99)00046-3

Zhao, Y., Xu, C., Xu, Y., Lin, J., Pang, Y., Liu, Z., & Shen, J. (2024). Mathematical exploration on control of bifurcation for a 3D predator–prey model with delay. *AIMS Mathematics,* 9(11), 29883–29915. https://doi.org/10.3934/math.20241442

Zhen, J., & Ma, Z. (2002). Stability for a competitive Lotka–Volterra system with delays. *Nonlinear Analysis: Theory, Methods & Applications,* 51(7), 1131–1142. https://doi.org/10.1016/S0362-546X(01)00883-0