Kontagora Journal of Science and Technology (KONJOST)
2025, Vol. 18, No.1-2, pp.48-55 | ISSN:1118-9673 (print)
https://doi.org/10.5281/zenodo.17881118

Research Article

Application of natural and Aboodh transforms to solve Caputo Fractional Differential Equations

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Abstract
Fractional calculus, which generalizes differentiation and integration to non-integer orders, is widely applied in modeling processes with memory and hereditary properties. However, solving Fractional Differential Equations (FDEs) remains analytically challenging. This study investigates the effectiveness of the Natural and Aboodh transforms for solving fractional differential equations defined in the Caputo sense. Using the operational properties of the transforms, fractional derivatives are simplified, the equations are reformulated in the transform domain, and explicit solutions are obtained through inverse transforms. Both homogeneous and nonhomogeneous cases are addressed, supported by illustrative examples. The transforms yield exact analytical solutions with reduced computational effort, offering accuracy and efficiency in handling fractional models compared to conventional approaches. The Natural and Aboodh transforms are shown to be reliable and elegant tools for solving complex fractional differential equations, making them valuable alternatives for applied sciences and engineering problems.

Keywords: Fractional Differential Equations; Natural Transform; Aboodh Transform; Laplace Transform; Analytical Solutions.

1. Introduction

Fractional Calculus deals with investigation and applications of integrals and derivatives of arbitrary order. The study of fractional order differential equations has gained significant attention in recent years due to their potential applications in various fields such as physics, engineering, biology, and finance. Despite their importance fractional order differential equations involve derivatives of non-integer order, which makes their analysis and solution quite challenging (Chen *et al*, 2022). Diverse integral transform methods such as Fourier, Laplace, Elzaki, Natural, Aboodh and Sumudu transforms have been employed by several researchers to handle this challenge (Aruldoss and Anusuya, 2020). Recently, Ahmad et al., (2024) present application of Laplace transform technique of variable order to the generalized caputo fractional model of second grade nanofluid, Noor et al., (2024) considered innovative solutions to the fractional diffusion equation using the Elzaki transform. Similarly, we reference the works in (Zayed, 2024; Rashid et al., 2024; Yavuz and Abdeljawad, 2020).

One of the key approaches to solving fractional order differential equations is through the use of analytical methods, which aim to find exact solutions without the need for numerical approximation. Some of these analytical methods which have received more attention by several researchers are the Natural transform method (Khan et al., 2008) and the Aboodh transform method (Aboodh, 2013).

The Natural transform is an extension of Laplace transform and is well-suited for solving initial value problems involving fractional derivatives while the Aboodh transform is a newer method that simplifies complex functions and is particularly useful for problems with fractional orders. These methods have been used successfully by (Alsaud et al., 2024) in their work “four-dimensional Natural Transform Adomian decomposition method and (3+1)-Dimensional Fractional Coupled Burgers’ equation” and by (Nadeem et al., 2024) in a paper titled “numerical investigation of two-dimensional fractional Helmholtz equation using Aboodh transform scheme”. The aim of this work, is to employed both the Natural and Aboodh transform methods to derive analytical solutions for some

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Received: 08/09/2025; **Accepted:** 24/09/2025; **Published:** 30/11/2025

homogeneous and inhomogeneous Fractional Differential Equations (FDEs).

Fractional differential equations remain challenging due to their complexity and limited exact solution methods. This study applies Natural and Aboodh transforms to address these gaps, aiming to obtain accurate analytical solutions. The approach is justified by its efficiency, simplicity, and potential to broaden applications across science and engineering.

2. Illustration and Properties

2.1 Fractional Differential Equation

The Caputo fractional differential equation of order α\alphaα (where n − 1< α≤n n-1 < n is given as:

$$\sum_{k=0}^{n} a_k D^{a_k} y(x) = f(x), \quad (1)$$

(Podlubny., and Kilbas, 2023).

Where;$y(x)$ is the unknown function of interest, $f(x)$ is a given function,$D^{\propto k}$ represents fractional derivatives of order $\propto_k$,a_k are coefficients, and$n$ is the order of the equation. Fractional integrals are considered as follows;

2.1.1. Riemann Fractional Integral

The Riemann fractional integral of order $\alpha > 0$ of a function $f(t) \in C_{\xi}, \xi \geq -1$ where α is non-negative, f(t) is piecewise continuous on (α, T) is defined as (Diethelm, 2010):

$$D^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^x (t-\tau)^{\alpha-1} f(\tau) d\tau \quad (2)$$

$$D^{o} f(t) = f(t). \quad (3)$$

The fractional derivatives using the definition of the fractional integral, let $n-1 < \alpha \leq n$ such that n is the smallest integer greater than on α , then the fractional derivatives of $f(t)$ is given as:

$$D_a^{\pi} f(t) = \frac{d^n}{dt^n}\left[D_{\alpha}^{n-\alpha} f(t)\right] = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau \quad (4)$$

2.1.2. Caputo Fractional Derivatives

The Caputo fractional derivative $D^{\propto}$ of a function $y(x)$ is defined as follows (Caputo, 1967):

$$D^{\propto} y(x) = \frac{1}{\Gamma(n-\propto)} \int_a^x (x-\xi)^{n-\propto-1} \frac{d^n y(\xi)}{d\xi^n} d\xi; \quad (5)$$

Where; 0<∝<1 is the order of the fractional derivative, $n-1<\propto<n$ such that n is the smallest integer greater than ∝, Γ denotes the gamma function, $\frac{d^n y(\xi)}{d\xi^n}$ denotes the nth derivative of $y(\xi)$, and a is a lower limit of integration, often set to zero.

2.2. Natural transform

Given a function $f(t)$ defined for $t \geq 0$, its natural transform $F(s)$ is defined as (Khan, 2008):

$$N\{f(t)\} = R(s,u) = \int_0^{\infty} e^{-st} f(ut) dt, \; u>0, s>0 \quad (6)$$

OR

$$F(s) = N\{f(t)\} = \frac{1}{u} \int_0^{\infty} e^{-\frac{s}{u}t} f(t) dt, \; u>0, \; s>0 \quad (7)$$

If $f^n(t)$ is the nth derivative of function $g(t)$, then Natural transform of $f^n(t)$ is given by (Belgacem, 2012),

$$N[f^n(t)] = R_n(s,u) = \frac{s^n}{u^n} R(s,u) - \sum_{k=0}^{n-1} \frac{s^{n-k-1}}{u^{n-k}} f^n(0), n \geq 1 \quad (8)$$

2.3. Aboodh transform

Given a function $f(t)$ defined for $t \geq 0$, its Aboodh transform $F(s)$ is defined as (Aboodh, 2013);

$$F(s) = A[f(t)] = \frac{1}{s} \int_0^{\infty} e^{-st} f(t) dt, \;\; t > 0 \; and s > 0 \quad (9)$$

If $f^n(t)$ is the nth derivative of function $f(t)$, then Aboodh transform of $f^n(t)$ is given by (Aboodh, 2013);

$$A[f^n(t)] = u^n \left[F(u) - \sum_{k=0}^{n-1} u^{k-2} f^n(0)\right], n \geq 1 \quad (10)$$

3. Basic Materials of the Methods

3.1 Natural Transform of Convolution

Let's denote the convolution of two functions $f(t)$ and $g(t)$ as $f(t)*g(t)$. The natural transform of the convolution $N\{f(t) * g(t)\}$ can be expressed as the product of the natural transforms of the individual functions:

$$N[f*g] = uF(s,u).G(s,u) \quad (11)$$

3.1.1. Natural Transform of Caputo Fractional Derivative

Let's denote the Caputo fractional derivative of a function $f(t)$ of order ∝ as $D_C^{\propto} f(t)$. The natural transform of this caputo fractional order derivative is denoted as $N\{D_C^{\propto} f(t)\}$.

The formula for the natural transform of the caputo fractional derivate can be expressed as;

$$N\{D_C^{\propto} f(t)\ \} = s^{\propto} N\{f(t)\} - s^{\propto-1} f(0) - s^{\propto-2} \frac{df(0)}{dt} - \dots - s\frac{d^{[\propto-1]}f(0)}{dt^{[\propto-1]}} \quad (12)$$

3.1.2. Aboodh Transform of Convolution

Let's denote the convolution of two functions *f (t)* and *g (t)* as *f(t)∗g(t).* The Aboodh transform of the convolution *A {f (t) ∗ g (t)}* can be expressed as the product of the Aboodh transforms of the individual functions:

$$A\,[f*g] = uF(s,u).G(s,u) \quad (13)$$

3.2. Aboodh Transform of Caputo Fractional Derivative

The Aboodh transform for Caputo time-fractional derivative of order $\boldsymbol{\beta}$ is given as follows (Ojo and Mahmudov, 2021] :

$$A\left[D_t^{\beta} Q(x,t);s\right] = s^{\beta} A[Q(x,t)] - \sum_{k=0}^{n-1} \frac{Q^k(x,0)}{S^{2-\beta+k}}, n-1<\beta\le n \quad (15)$$

3.3 Natural and Aboodh Transforms of basic Functions

Some of the Natural and Aboodh transforms of basic functions which are useful to this paper are considered in table1.

Table 1: Table of basic Functions and their Natural and Aboodh Transforms

F(t)	**N[f(t)]=F(s)**	**A[f(t)]=F(s)**
a	$N(a) = \frac{a}{s}$ $\quad a \ge 1$	$A(a) = \frac{a}{s^2}$ $\quad$ a≥ 1
t^n	$N(t^n) = \frac{u^n n!}{s^{n+1}}$ $\quad n \ge 1$	$A(t^n) = \frac{n!}{s^{n+2}}$ $\quad n \ge 1$
e^{at}	$N(e^{at}) = \frac{1}{s-au}$	$A(e^{at}) = \frac{1}{s(s-a)}$
Cosat	$N(\cos at) = \frac{s}{s^2+(au)^2}$	$A(\cos at) = \frac{1}{s^2+a^2}$
Sinat	$N(\sin at) = \frac{a}{s^2+(au)^2}$	$A(\sin at) = \frac{a}{s(s^2+a^2)}$
Coshat	$N(\cosh at) = \frac{s}{s^2-(au)^2}$	$A(\cosh at) = \frac{1}{s^2-a^2}$
Sinhat	$N(\sinh at) = \frac{a}{s^2-(au)^2}$	$A(\sinh at) = \frac{a}{s(s^2-a^2)}$
$f^n(t)$	$N[f^n(t)] = \frac{s^n}{u^n}R(s,u) - \sum_{k=0}^{n-1} \frac{s^{n-k-1}}{u^{n-k}} f^n(0)$ $\quad n \ge 1$	$A[f^n(t)] = u^n\left[F(u) - \sum_{k=0}^{n-1} u^{k-2} f^n(0)\right],$ $\quad$ n≥ 1

4. Applications

In this section, we employed both the Natural and Aboodh transform methods to derive analytical solutions for four examples of fractional order differential equations.

4.1. Example

Consider the inhomogeneous fractional ordinary differential equation as:

$D^2 f(t) + D^{\frac{3}{2}} f(t) + f(t) = t + 1 \quad t > 0$, Subject to the initial conditions (0) = *f ′* (0) = 1.

4.1.1.Natural transform on example 1

$$N[D^2 f(t)] + N\left[D^{\frac{3}{2}} f(t)\right] + N[f(t)] = N[t\} + N[1]$$

$$\frac{s^2}{u^2}R(s,u) - \frac{s}{u^2}f(0) - \frac{f'(0)}{u} + \frac{u^{\frac{1}{2}}}{s^{\frac{1}{2}}}\left(\frac{s^2}{u^2}R(s,u) - \frac{s}{u^2}f(0) - \frac{f'(0)}{u}\right) + R(s,u) = \frac{u}{s^2} + \frac{1}{s}$$

$$R(s,u)\left(\frac{s^2}{u^2} + \frac{s^{\frac{3}{2}}}{u^{\frac{3}{2}}} + 1\right) = \frac{u}{s^2} + \frac{1}{s} + \frac{s}{u^2} + \frac{1}{u} + \frac{s^{\frac{1}{2}}}{u^{\frac{3}{2}}} + \frac{s^{-\frac{1}{2}}}{u^{\frac{3}{2}}}$$

$$R(s,u)\left(\frac{u^{\frac{3}{2}}s^2 + u^2 s^{\frac{3}{2}} + u^{\frac{7}{2}}}{u^{\frac{7}{2}}}\right) = \frac{s+u}{s^2} + \frac{s+u}{u^2} + \frac{u^{\frac{1}{2}}s^{\frac{1}{2}} + u^{\frac{3}{2}}s^{-\frac{1}{2}}}{u^2}$$

$$R(s,u)\left(\frac{u^{\frac{3}{2}}s^2 + u^2 s^{\frac{3}{2}} + u^{\frac{7}{2}}}{u^{\frac{7}{2}}}\right) = \frac{s+u}{s^2} + \frac{s+u}{u^2} + \frac{(s+u)(s^{-\frac{1}{2}}u^{\frac{1}{2}})}{u^2}$$

$$R(s,u)\left(\frac{u^{\frac{3}{2}}s^2 + u^2 s^{\frac{3}{2}} + u^{\frac{7}{2}}}{u^{\frac{7}{2}}}\right) = (s+u)\left(\frac{1}{s^2} + \frac{1}{u^2} + \frac{s^{-\frac{1}{2}}u^{\frac{1}{2}}}{u^2}\right)$$

$$R(s,u) = (s+u)\left(\frac{u^2 + s^2 + s^{\frac{3}{2}}u^{\frac{1}{2}}}{s^2u^2}\right)\left(\frac{u^{\frac{7}{2}}}{u^{\frac{3}{2}}s^2 + u^2 s^{\frac{3}{2}} + u^{\frac{7}{2}}}\right)$$

$$R(s,u) = \frac{(s+u)}{s^2}\left(\frac{u^{\frac{11}{2}} + u^{\frac{7}{2}}s^2 + u^4 s^{\frac{3}{2}}}{u^{\frac{7}{2}}s^2 + u^4 s^{\frac{3}{2}} + u^{\frac{11}{2}}}\right)$$

$$R(s,u) = \frac{s+u}{s^2}$$

$$R(s,u) = \frac{s}{s^2} + \frac{u}{s^2}$$

$$R(s,u) = \frac{1}{s} + \frac{u}{s^2}$$

Applying the inverse Natural transform, gives

$$N^{-1}[R(s,u)] = N^{-1}\left[\frac{1}{s} + \frac{u}{s^2}\right]$$

$$f(t) = 1 + t$$

4.1.2.Aboodh transform on example 1

$$A[D^2 f(t)] + A\left[D^{\frac{3}{2}} f(t)\right] + A[f(t)] = A[t\} + A[1]$$

$$s^2 f(t) - f(0) - \frac{f'(0)}{s} + s^{\frac{3}{2}} f(t) - \frac{f(0)}{s^{\frac{1}{2}}} - \frac{f'(0)}{s^{\frac{3}{2}}} + f(t) = \frac{1}{s^3} + \frac{1}{s^2}$$

$$f(t)\left(s^2 + s^{\frac{3}{2}} + 1\right) = \frac{1}{s^3} + \frac{1}{s^2} + 1 + \frac{1}{s} + \frac{1}{s^{\frac{1}{2}}} + \frac{1}{s^{\frac{3}{2}}}$$

$$f(t)\left(s^2 + s^{\frac{3}{2}} + 1\right) = \frac{1 + s + s^3 + s^2 + s^{\frac{7}{2}} + s^{\frac{5}{2}}}{s^3}$$

$$f(t)\left(s^2 + s^{\frac{3}{2}} + 1\right) = \frac{s+1}{s^3} + \frac{s+1}{s} + \frac{s^{\frac{3}{2}} + s^{\frac{1}{2}}}{s^2}$$

$$f(t)\left(s^2 + s^{\frac{3}{2}} + 1\right) = (1+s)\left(\frac{1 + s^2 + s^{\frac{3}{2}}}{s^3}\right)$$

$$f(t) = (1+s)\left(\frac{1 + s^2 + s^{\frac{3}{2}}}{s^3}\right)\left(\frac{1}{\left(s^2 + s^{\frac{3}{2}} + 1\right)}\right)$$

$$f(t) = \frac{1+s}{s^3}$$

$$f(t) = \frac{1}{s^3} + \frac{s}{s^3}$$

$$f(t) = \frac{1}{s^3} + \frac{1}{s^2}$$

Applying the inverse Aboodh Transform, gives;

$$A^{-1}[f(t)] = A^{-1}\left[\frac{1}{s^3} + \frac{1}{s^2}\right]$$

$$f(t) = 1 + t$$

4.2. Example 2

Consider the fractional ordinary differential equation:

$$D^2y(\mathrm{t}) + \mathrm{D}^{\frac{3}{2}}\mathrm{y(t)} + \mathrm{y(t)} = \mathrm{t}^2 + 2 + 4\sqrt{\frac{\mathrm{t}}{\pi}}, 1 \le \mathrm{t} \le 2 \quad ,$$

Subject to the initial conditions y(0)=0.

4.2.1. Natural transform on example 2

$$N[D^2y(t)] + N\left[D^{\frac{3}{2}}y(t)\right] + N[y(t)] = N[\mathrm{t}^2] + N[2] + N\left[4\sqrt{\frac{\mathrm{t}}{\pi}}\right]$$

$$\frac{s^2}{u^2}y(s,u) - \frac{s}{u^2}y(0) + \frac{u^{\frac{1}{2}}}{s^{\frac{1}{2}}}\left(\frac{s^2}{u^2}y(s,u) - \frac{s}{u^2}y(0)\right) + y(s,u) = \frac{2u^2}{s^3} + \frac{2}{s} + \frac{2u^{\frac{1}{2}}}{s^{\frac{1}{2}}}$$

$$\frac{s^2}{u^2}y(s,u) + \frac{s^{\frac{3}{2}}}{u^{\frac{3}{2}}}y(s,u) + y(s,u) = \frac{2u^2}{s^3} + \frac{2}{s} + \frac{2u^{\frac{1}{2}}}{s^{\frac{1}{2}}}$$

$$\mathrm{y}(s,u)\left(\frac{s^2}{u^2} + \frac{s^{\frac{3}{2}}}{u^{\frac{3}{2}}} + 1\right) = \frac{2u^2}{s^3} + \frac{2}{s} + \frac{2u^{\frac{1}{2}}}{s^{\frac{1}{2}}}$$

$$y(s,u)\left(\frac{s^2 + s^{\frac{3}{2}}u^{\frac{1}{2}} + u^2}{u^2}\right) = \frac{2u^2 + 2s^2 + 2s^{\frac{3}{2}}u^{\frac{1}{2}}}{s^3}$$

$$y(s,u) = \left(\frac{2(u^2 + s^2 + s^{\frac{3}{2}}u^{\frac{1}{2}})}{s^3}\right)\left(\frac{u^2}{s^2 + s^{\frac{3}{2}}u^{\frac{1}{2}} + u^2}\right)$$

$$\mathrm{y}(s,u) = \frac{2u^2}{s^3}$$

Applying the inverse Natural transform, gives

$$N^{-1}[y(s,u)] = N^{-1}\left[\frac{2u^2}{s^3}\right]$$

$$\mathrm{y}(t) = t^2$$

4.2.2. Aboodh transform on example 2

$$\mathrm{A}[D^2y(t)] + A\left[D^{\frac{3}{2}}y(t)\right] + A[y(t)] = A[\mathrm{t}^2] + A[2] + A\left[4\sqrt{\frac{\mathrm{t}}{\pi}}\right]$$

$$s^2y(s) - y(0) + s^{\frac{3}{2}}y(s) - \frac{y(0)}{s^{\frac{1}{2}}} + y(s) = \frac{2}{s^4} + \frac{2}{s^2} + \frac{2}{s^{\frac{5}{2}}}$$

$$\mathrm{y}(s)\left(s^2 + s^{\frac{3}{2}} + 1\right) = \frac{2 + 2s^2 + 2s^{\frac{3}{2}}}{s^4}$$

$$\mathrm{y}(s)\left(s^2 + s^{\frac{3}{2}} + 1\right) = \frac{2(1 + s^2 + s^{\frac{3}{2}})}{s^4}$$

$$\mathrm{y}(s)\left(s^2 + s^{\frac{3}{2}} + 1\right) = \left(\frac{2(1 + s^2 + s^{\frac{3}{2}})}{s^4}\right)\left(\frac{1}{s^2 + s^{\frac{3}{2}} + 1}\right)$$

$$y(s) = \left[\frac{2}{s^4}\right]$$

Applying the inverse Aboodh Transform, gives;

$$A^{-1}[y(s)] = A^{-1}\left[\frac{2}{s^4}\right]$$

$$y(s) = t^2$$

4.3 Example 3

Consider the proble $D^2y(\mathrm{t}) + \mathrm{D}^{\frac{3}{2}}\mathrm{y(t)} + 2\mathrm{y(t)} = 2 \quad 1 \le \mathrm{t} \le 2$

Subject to the initial conditions y(0) =1

4.3.1.Natural transform on example 3

$$N[D^2y(t)] + N\left[D^{\frac{3}{2}}y(t)\right] + N[2y(t)] = N[2]$$

$$\frac{s^2}{u^2}y(s,u) - \frac{s}{u^2}y(0) + \frac{u^{\frac{1}{2}}}{s^{\frac{1}{2}}}\left(\frac{s^2}{u^2}y(s,u) - \frac{s}{u^2}y(0)\right) + 2y(s,u) = \frac{2}{s}$$

$$\frac{s^2}{u^2}y(s,u) - \frac{s}{u^2} + \frac{u^{\frac{1}{2}}}{s^{\frac{1}{2}}}\left(\frac{s^2}{u^2}y(s,u) - \frac{s}{u^2}\right) + 2y(s,u) = \frac{2}{s}$$

$$\frac{s^2}{u^2}y(s,u) + \frac{s^{\frac{3}{2}}}{u^{\frac{3}{2}}}y(s,u) + 2y(s,u) = \frac{2}{s} + \frac{s}{u^2} + \frac{s^{\frac{1}{2}}}{u^{\frac{3}{2}}}$$

$$y(s,u)\left(\frac{s^2}{u^2} + \frac{s^{\frac{3}{2}}}{u^{\frac{3}{2}}} + 2\right) = \frac{2}{s} + \frac{s}{u^2} + \frac{s^{\frac{1}{2}}}{u^{\frac{3}{2}}}$$

$$y(s,u)\left(\frac{s^2 + s^{\frac{3}{2}}u^{\frac{1}{2}} + 2u^2}{u^2}\right) = \frac{2u^2 + s^2 + s^{\frac{3}{2}}u^{\frac{1}{2}}}{su^2}$$

$$y(s,u) = \left(\frac{2u^2 + s^2 + s^{\frac{3}{2}}u^{\frac{1}{2}}}{su^2}\right)\left(\frac{u^2}{s^2 + s^{\frac{3}{2}}u^{\frac{1}{2}} + 2u^2}\right)$$

$$y(s,u) = \frac{1}{s}$$

Applying the inverse Natural transform, gives

$$N^{-1}[y(s,u)] = N^{-1}\left[\frac{1}{s}\right]$$

$y(t)$=1

4.3.2.Aboodh transform on example 3

$$A[D^2y(t)] + A\left[D^{\frac{3}{2}}y(t)\right] + A[2y(t)] = A[2]$$

$$s^2y(s) - y(0) - \frac{y'(0)}{s^2} + s^{\frac{3}{2}}y(s) - \frac{y(0)}{s^{\frac{1}{2}}} - \frac{y'(0)}{s^{\frac{5}{2}}} + 2y(s) = \frac{2}{s^2}$$

$$s^2y(s) - 1 + s^{\frac{3}{2}}y(s) - \frac{1}{s^{\frac{1}{2}}} + 2y(s) = \frac{2}{s^2}$$

$$y(s)\left(s^2 + s^{\frac{3}{2}} + 2\right) = \frac{2}{s^2} + 1 + \frac{1}{s^{\frac{1}{2}}}$$

$$y(s)\left(s^2 + s^{\frac{3}{2}} + 2\right) = \frac{2 + s^2 + s^{\frac{3}{2}}}{s^2}$$

$$y(s)\left(s^2 + s^{\frac{3}{2}} + 2\right) = \left(\frac{2 + s^2 + s^{\frac{3}{2}}}{s^2}\right)\left(\frac{1}{s^2 + s^{\frac{3}{2}} + 2}\right)$$

$$y(s) = \left[\frac{1}{s^2}\right]$$

Applying the inverse Aboodh Transform, gives;

$$A^{-1}[y(s)] = A^{-1}\left[\frac{1}{s^2}\right]$$

$$y(t) = 1$$

4.4. Example 4

Consider the linear fractional differential equation given as:

$D^{0.5}f(t) + f(t) = 0,\ 1 \le t \le 2$ Subject to $f(0) = 2$

4.4.1.Natural transform on example 4

$$N[D^{0.5}f(t)] + N[f(t)] = 0$$

$$s^{\frac{1}{2}}R(s,u) - s^{-\frac{1}{2}}f(0) + R(s,u) = 0$$

$$s^{\frac{1}{2}}R(s,u) - 2s^{-\frac{1}{2}} + R(s,u) = 0$$

$$s^{\frac{1}{2}}R(s,u) + R(s,u) = 2s^{-\frac{1}{2}}$$

$$(s^{\frac{1}{2}} + 1)R(s,u) = 2s^{-\frac{1}{2}}$$

$$R(s,u) = \frac{2s^{-\frac{1}{2}}}{s^{\frac{1}{2}}+1}$$

$$N^{-1}[R(s,u)] = N^{-1}\left(\frac{2s^{-\frac{1}{2}}}{s^{\frac{1}{2}}+1}\right)$$

$$f(t) = 2N^{-1}\left(\frac{s^{-\frac{1}{2}}}{s^{\frac{1}{2}}+1}\right)$$

$$recall\ N^{-1}\left[\frac{s^{\frac{1}{2}-\beta}}{s^{\frac{1}{2}}-\lambda}\right] = t^{\beta-1}E_{\frac{1}{2},\beta}\left(\lambda\, t^{\frac{1}{2}}\right)$$

Comparing we have;

$$-\frac{1}{2} = \frac{1}{2} - \beta\ \ thus\ \beta = 1\ and\ \lambda = -1$$

$$\therefore\ N^{-1}\left[\frac{s^{-\frac{1}{2}}}{s^{\frac{1}{2}}+1}\right] = 2t^{1-1}E_{\frac{1}{2},1}\left(-1\, t^{\frac{1}{2}}\right)$$

$$thus\ y = 2E_{\frac{1}{2},1}\left(-1\, t^{\frac{1}{2}}\right)$$

4.4.2.Aboodh transform on example 4

$$A[D^{0.5}f(t)]\ +\ A[f(t)] = 0$$

$$s^{\frac{1}{2}}f(s) - s^{-\frac{3}{2}}f(0) + f(s) = 0$$

$$s^{\frac{1}{2}}f(s) - 2s^{-\frac{3}{2}} + f(s) = 0$$

$$s^{\frac{1}{2}}f(s) + f(s) = 2s^{-\frac{3}{2}}$$

$$(s^{\frac{1}{2}}+1)f(s) = 2s^{-\frac{3}{2}}$$

$$f(s) = \frac{2s^{-\frac{3}{2}}}{s^{\frac{1}{2}}+1}$$

$$A^{-1}[f(s)] = A^{-1}\left(\frac{2s^{-\frac{3}{2}}}{s^{\frac{1}{2}}+1}\right)$$

$$f(t) = 2A^{-1}\left(\frac{s^{-\frac{3}{2}}}{s^{\frac{1}{2}}+1}\right)$$

$$recall\ A^{-1}\left[\frac{s^{-\frac{1}{2}-\beta}}{s^{\frac{1}{2}}-\lambda}\right] = t^{\beta-1}E_{\frac{1}{2},\beta}\left(\lambda\, t^{\frac{1}{2}}\right)$$

Comparing we have;

$$-\frac{3}{2} = -\frac{1}{2} - \beta\ \ thus\ \beta = 1\ and\ \lambda = -1$$

$$\therefore\ A^{-1}\left[\frac{s^{-\frac{3}{2}}}{s^{\frac{1}{2}}+1}\right] = 2t^{1-1}E_{\frac{1}{2},1}\left(-1\, t^{\frac{1}{2}}\right)$$

$$thus\ y = 2E_{\frac{1}{2},1}\left(-1\, t^{\frac{1}{2}}\right)$$

5. Conclusion

The Natural and Aboodh transform methods are powerful tools in applied mathematics and engineering and have been applied in solving homogeneous and inhomogeneous fractional differential equations. For the purpose of this research, the application of Natural and Aboodh transform is investigated to obtain an exact solution of some linear fractional order differential equations. The fractional order derivatives are described in the Caputo sense which obtained by Riemann-Liouville fractional integral operator. The solutions to some examples has show that both transforms are powerful and efficient techniques for obtaining analytic solution of inhomogeneous fractional order differential equations.

Abbreviations

FDEs Fractional Differential Equations

Acknowledgments

The authors sincerely acknowledge the contribution of Dr. K. I. Falade for his advice and proof reading of the manu-scripts.

Author Contributions

A. S. Raifu: Conceptualization, Methodology, Writing- original draft,
T. J. Kpanja: Writing - original draft, Formal Analysis
J. Balogun: Formal Analysis, Validation

Conflicts of Interest

The author(s) declare that they have no known competing financial interests, professional affiliations or personal relationships that could have appeared to influence the work reported in this paper.

Funding

None

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