Kontagora Journal of Mathematics(KJM)
Volume 1, Issue 1, September 2025
https://fuekjournals.org/index.php/KJM

Modelling the Equation of the Circular Restricted Three-Body Problem with Albedo Effect, Disk Of Dust And Variable Mass.

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DOI: https://doi.org/10.5281/zenodo.17366280

Abstract

The circular restricted Three-body problem (CR3BP) assumes two primaries of constant mass and an infinitesimal body moving in the plane described by the two primaries without the realistic perturbations. Studies have extended the circular restricted Three-body problem to include oblateness, radiation pressure and dissipative forces. Classical assumption considered the primaries as point mass with constant mass and neglected other perturbations that arise in realistic settings. There exist a gap in the literature on studies of comprehensive models that integrate multiple perturbations simultaneously. The CR3BP has been modified to realize new problems by incorporating the perturbations. However, little attention have been given to the combined effects of cluster belts, such as asteroid or dust belts, together with variable mass of the primaries. This work study the incorporation of oblateness, radiation pressure, variable mass and the gravitational influence of a cluster belt to model the equation of motion of the infinitesimal body under the influence of the perturbative effects. The numerical computation shows that the inclusion of perturbative forces significantly change the configuration as the equations of motion differs from the classical equations by the extra terms $\frac{\sigma^2}{4}x$ *and* $\frac{\sigma^2}{4}y$ *due to albedo effect, oblateness, disc of dust and variation in the mass of the third body. These results provide deeper insights into the dynamical behaviour of small celestial bodies and spacecraft in perturbed binary systems. The contributions of this work extend the theoretical framework of CR3BP thereby providing more realistic model for mission design*

Keywords: Albedo, Centrifugal force, Coriolis force, Oblate Spheroid, Perturbation, Radiation, Variable mass, Non-isotropic

Introduction

The study of the celestial mechanics/space dynamics has been fundamental to our understanding of planetary motion, spacecraft dynamics and astrophysical systems. The N-body problem which describes the mutual gravitational interactions of celestial bodies has been a central pursuit since Newton's principia (Newton, 1687). While exact analytical solution exist for two-body problem higher order cases generally require approximations or numerical treatment (szebehely, 1967).

The circular restricted three-body problem CR3BP represents a simplification in which two massive primaries revolve in circular orbits about their barycenter while a third body of infinitesimal mass moves under their gravitational influence without disturbing the primariess. Historically, the first specific three-body problem to receive extended study was the one involving the Moon, Earth, and the Sun (Wolfram, 2002). In an extended modern sense, a three-body problem is any problem in classical mechanics or quantum mechanics that models the motion of three particles by taking the initial positions and velocities (or momenta) of three-point masses and solving for their subsequent motion according to Newton's laws of motion and Newton's law of universal gravitation (Barrow-Green, 2008). The three-body problem is a special case of the n-body problem. Unlike two-body problems, no general closed-form solution exists (Barrow-Green, 2008), as the resulting dynamical system is chaotic for most initial conditions, and numerical methods are generally required. Its mass is supposed negligible. With this approximation, the study of the dynamics is reduced to the analysis of the motion of the third body which is a good approximation for most of the natural bodies in the Solar System.The general problem of finding a closed form solution for the motion of $N > 2$ bodies under their mutual gravitation has remained intractable but with specific assumptions, a handful of closed form solutions taking on different perturbations have been found. Unlike the case with two bodies, the three body problem has no simple general solution that can be written down to describe the motion for all time for any arbitrary initial conditions.

Researches had been carried out looking for a generalization of the problem of finding a system of solution for CR3BP, see El-Shaboury and Mustafa; 2013, Abouelmagd and El-Shaboury (2012), Abouelmagd and Sharaf (2013), Abouelmagd *et al*. (2016), Idrisi (2017a, 2017b), M. Javed Idrisi and M. Shahbaz Ullah (2017). The equations of motion in case of Albedo effect are derived. Tajudeen O. A. and Oni Leke (2024) derived the dynamical equations of the R3BP with perturbation on Poynting-Robertson (P-R) Drag force and variable masses. This derivation assumed variation for both primaries under the combined Mestschersky law (CML) and moved in the frame of the Gylden-Mestschersky equation (GME). Further, the bigger primary emits radiation force, which is a component of the radiation pressure and the P-R drag. It was observed that the P-R drag of the bigger primary depends on the mass parameter, radiation pressure, velocity of light and the mass variation constant.

Many scientists studied this model by taking different perturbations as different shapes of the primaries, the solar radiation pressure, the resonance, the variation of masses, the coriolis and centrifugal forces, the Yarkovsky effect, the Poynting-Robertson drags, Albedo effect etc. Yousuf and Kishor (2023) imposed the perturbation factors such as radiation pressure due to first oblate-radiating primary, albedo from second oblate primary. More other authors considered a situation where two primaries are oblate spheroid and surrounded by a disc of dusts in space with mass M_d, bigger oblate primary m_1 is a radiating body and second oblate primary m_2 produces albedo effect but neglected variable mass. While some author investigated the effect of Albedo on the motion of the infinitesimal body in the circular restricted three body problem with the variation of all masses but neglected a disc of dusts in space surrounded the masses. Thus, in this study we are conducting a research on a circular restricted three-body problem with perturbations and variable mass under the situation that:

(i) two primaries are oblate spheroid and surrounded by a disc of dusts in space with mass M_d,

(ii) bigger oblate primary m_1 is a radiating body,

(iii) second oblate primary m_2 produces albedo effect and

(iv) (iv) infinitesimal body is of variable mass.

With these combined effects; the research desires to formulate mathematical equations of motion governing the circular restricted three-body problem (CR3BP) under the perturbative parameters and variable mass using Jeans transformation technique, establish the criteria for the validation of the existence and uniqueness of solution of the equation formulated. This study can greatly enhance the knowledge and understanding of mathematical techniques presently used.

Equations of motion

In variable-mass systems, Newton's second law of motion cannot be directly applied because it is valid for constant mass systems only (Plastino and Muzio, 1992). Instead, a body whose mass m varies with time can be described by rearranging Newton's second law and adding term(s) to account for the momentum carried by the mass entering or leaving the system. The general equation of the variable-mass motion is written as

$$m\frac{d\vec{u}}{dt} = \vec{F}_{ext} - \vec{u}_{rel}\frac{dm}{dt}, \tag{1}$$

where $\vec{F}_{ext}$ is the net external force on the body, $\vec{u}_{rel} = \vec{v} - \vec{u}$ is the relative velocity of the escaping or incoming mass with respect to the body and $\vec{u}$ is the velocity of the body in the inertial frame, while $\vec{v}$ is the velocity of the escaping or incoming mass to the body. In astrodynamics, which deals with the mechanics of rockets, the term $\vec{u}_{rel}$ is often called the effective exhaust velocity. We impose the requirement that P_i is a body of mass m_i $(i = 1, 2, 3)$ with the position vectors r_i from the origin of the inertial frames XYZ and define, $\vec{\rho}_1 = \vec{r}_3 - \vec{r}_1$; $\vec{\rho}_2 = \vec{r}_3 - \vec{r}_2$; and $\vec{\rho}_3 = \vec{r}_2 - \vec{r}_1$. Now, if we assume the body P_3 has variable mass, ($m_3 = m_3(t)$). Then in the framework of the loss of mass being taken non-isotropic, one can apply Newton's second modified law in equation (1) to obtain the equations of motion for the infinitesimal body in the inertial frame in the case that the escaping or incoming mass occurs from n points of the body surface in the form

$$m_3\frac{d^2\vec{r}}{dt^2} = m_3\frac{\partial^2\vec{r}}{\partial t^2} + 2m_3\vec{\omega}\times\dot{\vec{r}} + m_3\vec{\omega}\times(\vec{\omega}\times\vec{r}) = -\frac{Gm_3m_1}{\rho_1^3}(\vec{r}-\vec{\rho}_1) - \frac{Gm_3m_2}{\rho_2^3}(\vec{r}-\vec{\rho}_2) - \dot{m}_3\sum_{i=1}^{n}\vec{u}_i \tag{2}$$

Equation (2) gives the equation of motion of the third body when its mass changes with time; (Varvoglis and Hadjidemetriou 2012).

We consider the motion of an infinitesimal mass m (such as small space objects, asteroid, spacecraft, satellite etc.) under the gravitational influence of two oblate masses (known as primaries) m_1 and m_2

such that $m << m_2 < m_1$ and a disc of dusts in space with mass M_d, which circumscribed the primaries and rotating about the common centre of mass of the system. The first oblate primary m_1 is a radiating body and second oblate primary m_2 is such that the incident radiations from the radiating oblate primary m_1 reflects back and hence m_2 produces albedo effect. Suppose, F_r and F_a are the radiation pressure force due to first radiating primary and albedo of second primary then mass reduction factor q (also called radiation parameter) and albedo Q_A and respectively defined (Ragos and Zagouras, 1993; Yousuf and Kishor, 2019, 2023). Let A_1 and A_2 be the oblateness coefficients of first and second primaries, which are respectively defined (McCuskey, 1963; Abouelmagd and Sharaf, 2013). The gravitational force exerted due to disc of the mass M_d on the infinitesimal mass defines a potential, which is expressed (Miyamoto and Nagai, 1975; Kushvah, 2008).

Then, if we assume that the infinitesimal mass m has variable mass, we will utilize the method used by (Abouelmagd and Mostafa, 2015) and hence the equations of motion of the infinitesimal mass of variable mass can be written as:

$$m\frac{d^2\vec{r}}{dt^2} = m\frac{\partial^2\vec{r}}{\partial t^2} + 2m\vec{\omega}\times\dot{\vec{r}} + m\vec{\omega}\times(\vec{\omega}\times\vec{r}) = -\frac{Gmm_1q\left(1+\frac{R^2A_1}{2\rho_1^2}\right)}{\rho_1^3}(\vec{r}-\vec{\rho}_1) - \frac{Gmm_2Q_A\left(1+\frac{R^2A_2}{2\rho_2^2}\right)}{\rho_2^3}(\vec{r}-\vec{\rho}_2) - n\dot{m}\dot{\vec{r}} - \frac{M_d\vec{r}}{m\left(r^2+T^2\right)^{\frac{3}{2}}} \quad (3)$$

That is

$$\ddot{\vec{r}} + 2\vec{\omega}\times\dot{\vec{r}} + n\frac{\dot{m}}{m}\dot{\vec{r}} = -\frac{Gm_1q\left(1+\frac{R^2A_1}{2\rho_1^2}\right)}{\rho_1^3}(\vec{r}-\vec{\rho}_1) - \frac{Gm_2Q_A\left(1+\frac{R^2A_2}{2\rho_2^2}\right)}{\rho_2^3}(\vec{r}-\vec{\rho}_2) - \vec{\omega}\times(\vec{\omega}\times\vec{r}) - \frac{M_d\vec{r}}{m\left(r^2+T^2\right)^{\frac{3}{2}}} \quad (4)$$

where the Coriolis force $\vec{F}_{cor}$ and centrifugal force $\vec{F}_{cf}$ can be expressed as:

$\vec{F}_{cor}$ ▯ $= -2\mathrm{m}\omega(\dot{y}, -\dot{x})$ and $\vec{F}_{cf}$ ▯ $= -\mathrm{m}\omega^2\dot{r}$

If the rotation frames rotates with the angular velocity ω, the relation between the inertial and rotating coordinates is governed by

$$\left.\begin{aligned} X &= x\cos\omega t - y\sin\omega t \\ Y &= x\sin\omega t + y\cos\omega t \\ Z &= z \end{aligned}\right\} \tag{5}$$

Then, the x–coordinate of

$$n\frac{\dot{m}}{m}\dot{\vec{r}} = n\frac{\dot{m}}{m}(\dot{x} - \omega y) \tag{6}$$

and the y –coordinate of

$$n\frac{\dot{m}}{m}\dot{\vec{r}} = n\frac{\dot{m}}{m}(\dot{y} + \omega x) \tag{7}$$

Then equations (4) can be expressed as:

$$\ddot{\vec{r}} - 2\omega(\dot{y}, -\dot{x}) + n\frac{\dot{m}}{m}(\dot{x} - \omega y, \dot{y} + \omega x) = -\frac{Gm_1 q\left(1 + \frac{R^2 A_1}{2\rho_1^2}\right)}{\rho_1^3}(\vec{r} - \vec{\rho}_1) - \frac{Gm_2 Q_A\left(1 + \frac{R^2 A_2}{2\rho_2^2}\right)}{\rho_2^3}(\vec{r} - \vec{\rho}_2) - \frac{M_d \vec{r}}{m(r^2 + T^2)^{\frac{3}{2}}} + \omega^2\vec{r} \tag{8}$$

where $\vec{\omega} = (0,\ 0,\ \omega)$ and

$$\rho_1^2 = (x-\mu R)^2 + y^2 , \qquad \rho_2^2 = (x+(1-\mu)R)^2 + y^2 \tag{9}$$

The components of equation (8) can be written in compact form as:

$$\ddot{x} - 2\omega\dot{y} + n\frac{\dot{m}}{m}(\dot{x} - \omega y) = \frac{\partial U}{\partial x}$$

(10)

$$\ddot{y} + 2\omega\dot{x} + n\frac{\dot{m}}{m}(\dot{y} + \omega x) = \frac{\partial U}{\partial y}$$

(11)

$$U(x,y) = \frac{\omega^2}{2}(x^2+y^2) + \frac{Gm_1 q\left(1+\frac{R^2A_1}{2\rho_1^2}\right)}{\rho_1} + \frac{Gm_2 Q_A\left(1+\frac{R^2A_2}{2\rho_2^2}\right)}{\rho_2} + \frac{2M_d}{m\sqrt{r^2+T^2}} \tag{12}$$

Equation (12) is the sum of the gravitational and centrifugal potential and belt and $\rho_1 = \sqrt{((x-\mu R)^2 + y^2)}$, $\rho_2 = \sqrt{((x+(1-\mu)R)^2 + y^2)}$ and $r = \sqrt{(x^2+y^2)}$ are the distances of the infinitesimal mass from primaries and from the origin, respectively.

Circular restricted three body problem has two natural scales: the distance R between masses m_1 and m_2, and the characteristic time of their orbital motion $\frac{1}{\omega}$. According to Jeans' transformation law and using the classical equation (4) we arrive at the following equations.

$$\frac{d^2x}{dt^2} - 2\omega\frac{dy}{dt} - \sigma(n-1)\frac{dx}{dt} + \sigma\omega(n-1)y = \frac{\partial\Omega}{\partial x} = -\frac{q(\sqrt{\eta})^3\left(1+\frac{\eta A_1}{2r_1^2}\right)}{r_1^3}(1-\mu)(x-\mu\sqrt{\eta}) -$$
$$\frac{Q_A(\sqrt{\eta})^3\left(1+\frac{\eta A_2}{2r_2^2}\right)}{r_2^3}\mu(x+(1-\mu)\sqrt{\eta}) - \frac{M_d x}{(r^2+T^2)^{\frac{3}{2}}} + \omega^2 x + \frac{\sigma^2(2n-1)}{4}x \tag{13}$$

$$\frac{d^2y}{dt^2}+2\omega\frac{dx}{dt}-\sigma(n-1)\frac{dy}{dt}-\sigma\omega(n-1)x=\frac{\partial\Omega}{\partial y}=-\frac{q\left(\sqrt{\eta}\right)^3\left(1+\frac{\eta A_1}{2r_1^2}\right)}{r_1^3}(1-\mu)y-$$
$$\frac{Q_A\left(\sqrt{\eta}\right)^3\left(1+\frac{\eta A_2}{2r_2^2}\right)}{r_2^3}\mu y-\frac{M_d y}{\left(r^2+T^2\right)^{\frac{3}{2}}}+\omega^2 y+\frac{\sigma^2(2n-1)}{4}y \quad (14)$$

Equations (13) and (14) can be rewritten in a compact form as

$$\ddot{x}-2\omega\dot{y}-\sigma(n-1)\dot{x}+\sigma\omega(n-1)y=\frac{\partial\Omega}{\partial x}$$

(15)

$$\ddot{y}+2\omega\dot{x}-\sigma(n-1)\dot{y}-\sigma\omega(n-1)x=\frac{\partial\Omega}{\partial y}$$

(16) (15) and (16) represented the equation of motion of the restricted three body problem in the sense that the variation of the third body is non-isotropic when the variation is from the whole surface. If we assume that the variation of the mass originates from one point ($n=1$) then (15) and (16) become

$$\ddot{x}-2\omega\dot{y}=\frac{\partial\Omega}{\partial x} \quad (17)$$

$$\ddot{y}+2\omega\dot{x}=\frac{\partial\Omega}{\partial y} \quad (18)$$

where

$$\Omega(x,y)=\left(\frac{\omega^2}{2}+\frac{\sigma^2}{8}\right)\left(x^2+y^2\right)+\left(\sqrt{\eta}\right)^3\left(\frac{q(1-\mu)\left(1+\frac{\eta A_1}{2r_1^2}\right)}{r_1}+\frac{Q_A\mu\left(1+\frac{\eta A_2}{2r_2^2}\right)}{r_2}\right)+$$
$$\frac{2M_d}{\left(r^2+T^2\right)^{\frac{1}{2}}} \quad (19)$$

That is

$$\Omega(x,y)=\left(\frac{\omega^2}{2}+\frac{\sigma^2}{8}\right)\left((1-\mu)r_1^2+\mu r_2^2-\eta\mu(1-\mu)\right)+\left(\sqrt{\eta}\right)^3\left(\frac{q(1-\mu)\left(1+\frac{\eta A_1}{2r_1^2}\right)}{r_1}+\frac{Q_A\mu\left(1+\frac{\eta A_2}{2r_2^2}\right)}{r_2}\right)+\frac{2M_d}{\left(r^2+T^2\right)^{\frac{1}{2}}} \tag{20}$$

$$\frac{\partial\Omega}{\partial x}=\omega^2x+\frac{\sigma^2}{4}x-\left(\sqrt{\eta}\right)^3\left(\frac{q(1-\mu)\left(1+\frac{3\eta A_1}{2r_1^2}\right)}{r_1^3}\left(x-\sqrt{\eta}\mu\right)+\frac{Q_A\mu\left(1+\frac{3\eta A_2}{2r_2^2}\right)}{r_2^3}\left(x+\sqrt{\eta}(1-\mu)\right)\right)-\frac{M_d x}{\left(r^2+T^2\right)^{\frac{3}{2}}} \tag{21}$$

and

$$\frac{\partial\Omega}{\partial y}=\omega^2y+\frac{\sigma^2}{4}y-\left(\sqrt{\eta}\right)^3\left(\frac{q(1-\mu)\left(1+\frac{3\eta A_1}{2r_1^2}\right)}{r_1^3}y+\frac{Q_A\mu\left(1+\frac{3\eta A_2}{2r_2^2}\right)}{r_2^3}y\right)-\frac{M_d y}{\left(r^2+T^2\right)^{\frac{3}{2}}} \tag{22}$$

Equations (17) and (18) are different from the classical equations (3) or (4) by the extra terms $\frac{\sigma^2}{4}x$ and $\frac{\sigma^2}{4}y$ due to albedo effect, oblateness, disc of dust and variation in the mass of the third body.

Existence and Uniqueness of Solution

Here, the equations (17) and (18) are considered in the form:

$$\ddot{x}=A_0x+B_0+2\omega\dot{y} \tag{23}$$

$$\ddot{y}=A_0y-2\omega\dot{x} \tag{24}$$

where

$$A_0 = \left(\omega^2 + \frac{\sigma^2}{4} - \left(\sqrt{\eta}\right)^3 \left(\frac{q(1-\mu)\left(1 + \frac{3\eta A_1}{2r_1^2}\right)}{r_1^3} + \frac{Q_A \mu \left(1 + \frac{3\eta A_2}{2r_2^2}\right)}{r_2^3} \right) - \frac{M_d}{\left(r^2 + T^2\right)^{\frac{3}{2}}} \right) \tag{25}$$

and

$$B_0 = \left(\sqrt{\eta}\right)^3 \left(\frac{q(1-\mu)\left(1 + \frac{3\eta A_1}{2r_1^2}\right)}{r_1^3} \sqrt{\eta}\mu + \frac{Q_A \mu \left(1 + \frac{3\eta A_2}{2r_2^2}\right)}{r_2^3} \sqrt{\eta}(1-\mu) \right) \tag{26}$$

Let

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} t \\ x \\ \dot{x} \\ y \\ \dot{y} \end{pmatrix}$$

(27)

Differentiating (27) with respect to t will lead to

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \\ x_5' \end{pmatrix} = \begin{pmatrix} 1 \\ \dot{x} \\ \ddot{x} \\ \dot{y} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} 1 \\ x_3 \\ A_0 x_2 + B_0 + 2\omega x_5 \\ x_5 \\ A_0 x_4 - 2\omega x_3 \end{pmatrix} \tag{28}$$

Satisfying the initial conditions

$$\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \\ x_5(0) \end{pmatrix} = \begin{pmatrix} 0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$$

(29)

$0 < x_1 < \infty$, where a_1, a_2, a_3 and a_4 are guessed values.

Suppose $|x| \le k_1,\ |\dot{x}| \le k_2,\ |y| \le k_3,\ |\dot{y}| \le k_4,\ |A| \le k_5,\ |B| \le k_6$, where $\eta,\ \omega,\ \sigma,\ M_d,\ T,\ q,\ Q_A,\ A_1,\ A_2,\ \mu$ are real constants and $0 \le t < \infty$. Then the equations (17) and (18) have a unique solution. We now show that our problems satisfy the hypothesis of Derrick and Grossman (1976). We establish the proof as thus:

Proof: Let

$$z = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

Then

$$z' = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \\ x_5' \end{pmatrix} = f(z) = \begin{pmatrix} 1 \\ x_3 \\ A_0 x_2 + B_0 + 2\omega x_5 \\ x_5 \\ A_0 x_4 - 2\omega x_3 \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2, x_3, x_4, x_5) \\ f_2(x_1, x_2, x_3, x_4, x_5) \\ f_3(x_1, x_2, x_3, x_4, x_5) \\ f_4(x_1, x_2, x_3, x_4, x_5) \\ f_5(x_1, x_2, x_3, x_4, x_5) \end{pmatrix}$$

It would be further shown that the partial derivatives $\frac{\partial f_j}{\partial x_i},\ i,\ j = 1,\ 2, \ldots, 5$ are bounded and satisfying the interval.

Clearly,

$$\left|\frac{\partial f_1}{\partial x_i}\right| = 0, \text{ for } i = 1,2,\ldots 5$$

$$\left|\frac{\partial f_2}{\partial x_3}\right| = 1, \quad \left|\frac{\partial f_2}{\partial x_i}\right| = 0, \text{ for } i = 1,2,4,5$$

$$\left|\frac{\partial f_3}{\partial x_2}\right| = |A_0| \le k_5, \quad \left|\frac{\partial f_3}{\partial x_5}\right| = |2\omega| = 2\omega, \quad \left|\frac{\partial f_3}{\partial x_i}\right| = 0, \text{ for } i = 1,3,4$$

$$\left|\frac{\partial f_4}{\partial x_5}\right| = 1, \quad \left|\frac{\partial f_4}{\partial x_i}\right| = 0, \text{ for } i = 1,2,\ldots,4$$

$$\left|\frac{\partial f_5}{\partial x_3}\right| = |-2\omega| \le 2\omega, \quad \left|\frac{\partial f_5}{\partial x_4}\right| = |A_0| \le k_5, \quad \left|\frac{\partial f_5}{\partial x_i}\right| = 0, \text{ for } i = 1,2,5$$

Let $K = \max\{\, 0, 1, k_5, 2\omega \,\} < \infty$. Therefore the partial derivatives $\frac{\partial f_j}{\partial x_i}, \; i, j = 1, 2, \ldots, 5$ are continuous and bounded.

Hence by Derrick and Grossman (1976) theorem, equations (17) and (18) have a unique solution. This completes the proof.

Results

A mathematical model describing the motion of infinitesimal mass with variable mass around oblate primaries with one radiating and the other having albedo effect was formulated. The main objective of this study was to assess the impact of the perturbations on the equation of motion of infinitesimal mass with variable mass. It was found that the inclusion of perturbative forces significantly change the configuration as the equations of motion differs from the classical equations by the extra terms $\frac{\sigma^2}{4}x$ and $\frac{\sigma^2}{4}y$ due to albedo effect, oblateness, disc of dust and variation in the mass of the third body. These results provide deeper insights into the dynamical behaviour of small celestial bodies and spacecraft in perturbed binary systems. To say that the model is well posed, the criteria for the existence and uniqueness of solution of the model was established using Lipschitz continuity approach. This is to show that the solutions of model formulated depend continuously on the initial and boundary conditions. Further research is required on the existence, location and stability of the equilibrium points as the perturbation parameters varies.

Conclusion

The circular restricted three-body problem is studied in the context of infinitesimal body having variable mass changes according to Jeans' law. The equation of motion is derived when loss of mass is non-isotropic and under the influence of perturbations in the form of radiation due to oblate-radiating first primary, albedo of oblate second primary, oblateness of both the primaries and presence of the disc. The main objective of this study was to assess the impact of the perturbations on the dynamical equation motion of infinitesimal mass with variable mass. To say that the model is well posed, the criteria for the existence and uniqueness of solution of the model was established using Lipschitz continuity approach. This is to show that the solutions of the model formulated depend continuously on the initial and boundary conditions.

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