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Upper Estimates for Coefficients Bounds and Fekete-Szegö Functional of a Subclass of Univalent Functions Defined by Salagean Differential Operator

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Abstract

In this work, coefficient bounds and the Fekete-Szegö functional for functions analytic in the unit disk $\Delta = \{z \in \mathbb{C}: |z| < 1\}$ and belonging to the class

$$A_n^\tau(\beta) = \left\{ f \in \mathcal{A}: \left[(1-\tau)\frac{D^n f(z)}{z} + \tau \frac{D^{n+1} f(z)}{D^n f(z)} \right] \prec \beta(z) \right\}$$

where $z \in \Delta$, $\tau \in [0,1]$, $n \in \mathbb{N}_0$ and $\beta(z)$ is analytic in Δ such that $\Re\beta(z) > 0$, $\beta(0) = 1$ and $\beta'(z) > 0$ and maps Δ onto a region starlike with respect to 1 and symmetric with respect to the real axis were established.

Keywords: Analytic functions, subordination, Salagean differential operator, Ma-Minda function

Introduction

An interesting aspect of complex analysis is Geometric Function Theory (GFT), which focuses on geometric properties of analytic functions that has applications in different areas of mathematics, such as mathematical physics, conformal mappings and special functions.

Let $\mathcal{A}$ denotes the set of functions that are analytic in the unit disk

$$\Delta = \{z \in \mathbb{C}: |z| < 1\},$$

And let $\mathcal{S}$ denotes a subclass of $\mathcal{A}$ consisting of functions that are analytic-univalent in the unit disk Δ and has the series form

$$f(z) = z + \sum_{k=2}^{\infty} a_k\, z^k, \qquad (1)$$

satisfying the conditions: $f(0) = 0$, $f'(0) = 1$ and $z \in \Delta$.

In 1916, Bieberbach stated that for functions $f \in \mathcal{S}$, the modulus of the second coefficient is less or equal to 2, i.e $|a_2| \leq 2$ [7]. He went on to conjectured that $|a_k| \leq k$ which is known as Bieberbach conjecture. This conjecture, which was a long-standing problem in geometric function theory was eventually proven by Louis De-Branges in 1985 [6]. Duren [7] emphasised that coefficient problem involves determining $(a_1, a_2, \dots, a_k)$ corresponding to functions in $\mathcal{S}$.

Considerable attention has been devoted to estimating upper bounds for coefficients and the Fekete-Szegö functional of functions belonging to various well-known subclasses of $\mathcal{S}$. These include starlike, convex, close-to-starlike, close-to-convex functions, as well as generalisation of these subclasses defined via differential, integral and q-differential operators (See [2, 3, 8, 9, 10,11,12]).

Let f and g be analytic functions $f \prec g$, if there exists a function,

$$\omega(z) = c_1 z + c_2 z^2 + c_3 z^3 + \cdots, \quad (|\omega(z)| < 1, \quad \omega(0) = 0, \quad \text{and} \quad z \in \Delta), \qquad (2)$$

such that $f(z) = g(\omega(z))$. If g is univalent inΔ, then

$$f \prec g \quad \text{if and only if} \quad f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta$$

Definition 1. A function $p \in \mathcal{P}$ is a function with positive real part (Caratheodory function) if

$$p(z) = 1 + \sum_{m=1}^{\infty} p_m z^m, \quad \Re p(z) > 0, \quad p(0) = 1 \quad \text{and} \quad z \in \Delta. \tag{3}$$

Möbius function is an extremal function in class$\mathcal{P}$,

$$M_0(z) = \frac{1+z}{1-z} = 1 + 2\sum_{m=1}^{\infty} z^m, \quad z \in \Delta. \tag{4}$$

Definition 2. A function $\omega \in \mathcal{W}$ is a Schwarz function if

$$\omega(z) = c_1 z + c_2 z^2 + c_3 z^3 + \cdots, \quad (|\omega(z)| < 1, \quad \omega(0) = 0, \quad \text{and} \quad z \in \Delta).$$

It has been established that $\omega(z)$ and $p(z)$ are interrelated by

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)} \Longrightarrow \omega(z) = \frac{p(z)-1}{p(z)+1}, \quad (z \in \Delta) \tag{5}$$

Substituting (3) into (5) and with further evaluation gives,

$$\omega(z) = \frac{1}{2}\left(p_1 z + \left(p_2 - \frac{p_1^2}{2}\right) z^2 + \left(p_3 - p_1 p_2 + \frac{p_1^3}{4}\right) z^3 + \cdots\right). \tag{6}$$

In 1992, Ma-Minda [9] introduced an analytic function $\beta(z)$ by combining various kinds of functions in $\mathcal{P}$,

$$\beta(z) = 1 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots, \quad (b_1 > 0, \quad b_m \in \mathbb{R}, \quad z \in \Delta). \tag{7}$$

Here $\Re\beta(z) > 0$, $\beta(0) = 1$, $\beta'(0) = 1$, $\beta(z)$ maps Δ on a region starlike with respect to 1 and symmetric with respect to the real axis. Putting (6) into (7), simplifying further gives

$$\beta(\omega(z)) = 1 + \frac{b_1}{2} p_1 z + \left(\frac{b_1}{2}\left(p_2 - \frac{p_1^2}{2}\right) + \frac{b_2}{4} p_1^2\right) z^2$$
$$+ \left(\frac{b_1}{2}\left(p_3 - p_1 p_2 + \frac{p_1^3}{4}\right) + \frac{b_2}{2} p_1 \left(p_2 - \frac{p_1^2}{2}\right) + \frac{b_3}{8} p_1^3\right) z^3 + \cdots \tag{8}$$

Definition 3. Let $f \in \mathcal{A}$, then Salagean differential Operator $D^n f(z) : \mathcal{A} \to \mathcal{A}$ is defined by

$$D^0f(z)=f(z),\quad D^1f(z)=zf'(z),\quad \ldots \quad D^nf(z)=D\left(D^{n-1}f(z)\right)=z+\sum_{k=2}^{\infty}k^n\, a_k z^k,\quad n \in \mathbb{N}_0 \tag{9}$$

Bello [7], in one of her works obtained some properties of a class of analytic functions defined by Salagean differential operator.

2 Applicable Lemmas

Lemma 1 [4]. Let $p \in \mathcal{P}$ as defined in(3). Then

$$|p_m| \leq 2 \quad \forall \quad m \in \mathbb{N} \tag{10}$$

Lemma 2 [4]. Let $p \in \mathcal{P}$. Then

$$\left|p_2-\Lambda\frac{p_1^2}{2}\right| \leq \begin{cases} 2(1-\Lambda), & \text{if } \Lambda \leq 0 \\ 2, & \text{if } 0 \leq \Lambda \leq 2 \\ 2(\Lambda-1), & \text{if } \Lambda \geq 2 \\ 2\max[1,|1-\Lambda|], & \text{if } \Lambda \in \mathbb{C} \end{cases} \tag{11}$$

Lemma 3[1] Let $p \in \mathcal{P}$. Then

$$|up_1^3-vp_1p_2+wp_3| \leq 2|u|+2|v-2u|+2|u-v+w| \tag{12}$$

Lemma 4 [10]. Let $p \in \mathcal{P}$. Then

$$|p_{i+j}-\mu p_i p_j| \leq \begin{cases} 2, & \text{if } 0 \leq \mu \leq 1 \\ 2|2\mu-1|, & \text{elsewhere} \end{cases} \tag{13}$$

3 Main results

A function $f \in \mathcal{A}$ is in $A_n^{\tau}(\beta)$, if the subordination condition

$$\left[(1-\tau)\frac{D^nf(z)}{z}+\tau\frac{D^{n+1}f(z)}{D^nf(z)}\right] \prec \beta(z) \tag{14}$$

holds for $z \in \Delta$, $0 \leq \tau \leq 1$, $\beta(z)$ is Ma-Minda function (7) and $D^nf(z)$ is Salagean differential operator.

Remark 1. If $\tau=0$, $n=0$ and $\beta(z)=M_0(z)$, then

$$\frac{f(z)}{z} \prec M_0(z),$$

is a geometric condition for Yamaguchi functions [11].

Remark 2. If $\tau = 1, n = 0$ and $\beta(z) = M_0(z)$, then

$$\frac{zf'(z)}{f(z)} \prec M_0(z),$$

is a geometric condition for starlike functions.

In this work, the upper estimates for initial coefficient bounds and the Fekete-Szegö functional of functions in $A_n^\tau(\beta)$ were established.

Theorem1. Let $f \in A_n^\tau(\beta)$, then

$$\begin{aligned}
|a_2| &\le \frac{b_1}{2^n}\\
|a_3| &\le \frac{(2^n+\tau)b_1 + 2^n|b_2|}{2^n \cdot 3^n A}\\
|a_4| &\le \frac{b_1(A + 3A(A-1)b_1 + 2(A-1)^2 b_1) + 2^n|b_2|(5A-3)}{2^{2n}AB}\\
|a_5| &\le \frac{2b_1 - 3b_2 + 9b_3}{5^n C} + \frac{(A-1)[4b_1^2 + 8Db_1^2 + 4D^2]}{2\cdot 3^n \cdot 5^n A^2 C} + \frac{2(A-1)b_1}{5^n BC}\left[2b_1 + 4\left(b_2 + \frac{3}{2}b_1^2\right) + \frac{3(A-1)\mathrm{D}}{2^{-(1+n)}}\right]\\
&\quad + \frac{(A-1)b_1^4}{5^n C} + \frac{5(A-1)[2b_1^3 + 4Db_1]}{2^{2-n}5^n AC}, \qquad (15)
\end{aligned}$$

where $A = (1+\tau)$, $B = (1+2\tau)$, $C = (1+3\tau)$ and $D = b_1 + (A-1)\frac{b_1^2}{2^n}$.

Proof. Let $f \in A_n^\tau(\beta)$. By subordination, there exists a function$\omega(z)$, $\omega(0) = 0$ and $|\omega(z)| < 1$ such that

$$\left[(1-\tau)\frac{D^n f(z)}{z} + \tau\frac{D^{n+1}f(z)}{D^n f(z)}\right] = \beta\big(\omega(z)\big) \qquad (16)$$

Substituting (9) into LHS of (16) and simplifying further gives

$$
\begin{aligned}
&1+2^n a_2 z+\left((1+\tau)3^n a_3-\tau 2^n a_2^2\right)z^2+\left((1+2\tau)4^n a_4-\tau 2^n 3^{n+1}a_2a_3+23^n a_3^2\right)z^3\\
&\quad+\left((1+3\tau)5^n a_5-\tau 2^{3n+2}a_2a_4-\tau^2\cdot 3^{2n}a_2^3+\tau 5\cdot 2^{2n}3^n a_2^2 a_3-\tau^2 4^n a_4^2\right)z^4\\
&\quad+\cdots \qquad (17)
\end{aligned}
$$

Also, replacing RHS of (16) by (8), we have

$$
\begin{aligned}
&1+\frac{b_1}{2}p_1 z+\left(\frac{b_1}{2}(p_2-\frac{p_1^2}{2})+\frac{b_2}{4}p_1^2\right)z^2+\left(\frac{b_1}{2}(p_3-p_1p_2+\frac{p_1^3}{4})+\frac{b_2}{2}p_1(p_2-\frac{p_1^2}{2})+\frac{b_3}{8}p_1^3\right)z^3\\
&\quad+\cdots \quad (18)
\end{aligned}
$$

Comparing coefficients of z in (17) and (18) gives the following equations

$$2^n a_2=\frac{b_1p_1}{2} \qquad (19)$$

$$\left((1+\tau)3^n a_3-\tau 2^n a_2^2\right)=\frac{b_1}{2}\left(p_2-\frac{p_1^2}{2}\right)+\frac{b_2}{4}p_1^2 \qquad (20)$$

$$
\begin{aligned}
&\left((1+2\tau)4^n a_4-\tau 2^n 3^{n+1}a_2a_3+23^n a_3^2\right)\\
&=\frac{b_1}{2}\left(p_3-p_1p_2+\frac{p_1^3}{4}\right)+\frac{b_2}{2}p_1\left(p_2-\frac{p_1^2}{2}\right)+\frac{b_3}{8}p_1^3 \qquad (21)
\end{aligned}
$$

$$
\begin{aligned}
&((1+3\tau)5^n a_5-\tau 2^{3n+2}a_2a_4-\tau^2\cdot 3^{2n}a_2^3+\tau 5\cdot 2^{2n}3^n a_2^2a_3-\tau^2 4^n a_4^2)z^4\\
&=\frac{b_1}{2}\left(p_4-p_1p_3+\frac{3p_1^2p_2}{4}-\frac{p_2^2}{2}-\frac{p_1^4}{8}\right)+\frac{b_2}{4}\left(p_1p_3-p_1^2p_2+\frac{p_1^4}{2}-p_1^3p_2+p_1^2p_2^2\right)+\frac{3b_3}{8}\left(p_1^2p_2-\frac{p_1^4}{2}\right) \quad (22)
\end{aligned}
$$

From (19), (20), (21) and (22), a_2, a_3, a_4 and a_5 were obtained respectively after some calculations.

$$a_2=\frac{b_1p_1}{2^{n+1}} \qquad (23)$$

$$a_3=\frac{b_1}{2\cdot 3^n(1+\tau)}\left(p_2-\frac{p_1^2}{2}\right)+\frac{p_1^2}{4\cdot 3^n(1+\tau)}\left(b_2+\tau\frac{b_1^2}{2^n}\right) \qquad (24)$$

$$
\begin{aligned}
a_4&=\frac{b_1}{2^{2n+1}(1+2\tau)}\left(p_3-p_1p_2+\frac{p_1^3}{4}\right)+\left(\frac{b_2p_1}{2^{2n+1}(1+2\tau)}+\frac{3\tau b_1^2p_1}{2^{2n+2}(1+2\tau)}\right)\left(p_2-\frac{p_1^2}{2}\right)\\
&\quad+\frac{3\tau p_1^2}{2^{n+2}(1+\tau)(1+2\tau)}\left(b_2+\tau\frac{b_1^2}{2^n}\right) \qquad (25)
\end{aligned}
$$

$$\begin{aligned}(1+3\tau)a_5 = {}& \frac{b_1}{2}(p_4 - p_1p_3 + \frac{3}{4}p_1^2p_2 - \frac{p_2^2}{2} - \frac{p_1^4}{8}) + \frac{b_2}{4}(p_1p_3 - p_1^2 + p_2 + \frac{p_1^4}{2} - p_1^3p_2 + p_1^2p_2^2) \\ & + \frac{3b_3}{8}(p_1^2p_2 - \frac{p_1^4}{2}) \\ & + \frac{\tau}{2\cdot 3^n(1+\tau)^2}\left(b_1^2(p_2 - \frac{p_1^2}{2})^2 + b_1^2p_1^2(p_2 - \frac{p_1^2}{2})(b_2 + \tau\frac{b_1^2}{2^n}) + \frac{p_1^4}{4}(b_2 + \tau\frac{b_1^2}{2^n})^2\right) \\ & + \frac{\tau b_1p_1}{(1+\tau)}\left(b_1(p_3 - p_1p_2 + \frac{p_1^3}{4}) + (b_2p_1 + \frac{3b_1^2p_1}{2})(p_2 - \frac{p_1^2}{2}) + \frac{3\tau p_1^2}{2^{1-n}(1+\tau)}(b_2 + \tau\frac{b_1^2}{2^n})^2\right) \\ & - \frac{5\tau}{2^{2-n}(1+\tau)}\left(b_1^3p_1^2(p_2 - \frac{p_1^2}{2}) + \frac{b_1p_1^3}{2}(b_2 + \tau\frac{b_1^2}{2^n})\right) + \frac{\tau b_1^4p_1^4}{2^4} \qquad (26)\end{aligned}$$

Taking bound on (23) and applying lemma 1 gives

$$|a_2| \le \frac{b_1}{2^n}. \qquad (27)$$

Also, taking bound on (24) gives

$$|a_3| = \left|\frac{b_1}{2\cdot 3^n(1+\tau)}\left(p_2 - \frac{p_1^2}{2}\right) + \frac{p_1^2}{4\cdot 3^n(1+\tau)}\left(b_2 + \tau\frac{b_1^2}{2^n}\right)\right|, \qquad (28)$$

With application of Lemma 2 (when Λ is 1) and Lemma 1 on (28), we have

$$|a_3| \le \frac{(2^n + \tau)b_1 + 2^n|b_2|}{2^n\cdot 3^n A} \qquad (29)$$

Furthermore, taking bound on (25) gives

$$\begin{aligned}|a_4| = {}& \left|\frac{b_1}{2^{2n+1}(1+2\tau)}\left(p_3 - p_1p_2 + \frac{p_1^3}{4}\right) + \left(\frac{b_2p_1}{2^{2n+1}(1+2\tau)} + \frac{3\tau b_1^2p_1}{2^{2n+2}(1+2\tau)}\right)\left(p_2 - \frac{p_1^2}{2}\right)\right. \\ & \left. + \frac{3\tau p_1^2}{2^{n+2}(1+\tau)(1+2\tau)}\left(b_2 + \tau\frac{b_1^2}{2^n}\right)\right|, \qquad (30)\end{aligned}$$

Application of Lemma 3 (when $u = \frac{1}{4}$, $v = 1$, $w = 1$), Lemma 2 (when $\Lambda = 1$) and Lemma 1 results to

$$|a_4| \le \frac{b_1(A + 3A(A-1)b_1 + 2(A-1)^2b_1) + 2^n|b_2|(5A-3)}{2^{2n}AB} \qquad (31)$$

Finally, taking bound on (26) gives

$$
\begin{aligned}
|(1+3\tau)a_5| = \Bigg| & \frac{b_1}{2}\left(p_4 - p_1p_3 + \frac{3}{4}p_1^2p_2 - \frac{p_2^2}{2} - \frac{p_1^4}{8}\right) + \frac{b_2}{4}\left(p_1p_3 - p_1^2 + p_2 + \frac{p_1^4}{2} - p_1^3p_2 + p_1^2p_2^2\right) \\
& + \frac{3b_3}{8}(p_1^2p_2 - \frac{p_1^4}{2}) \\
& + \frac{\tau}{2\cdot 3^n(1+\tau)^2}\left(b_1^2(p_2 - \frac{p_1^2}{2})^2 + b_1^2p_1^2(p_2 - \frac{p_1^2}{2})(b_2 + \tau\frac{b_1^2}{2^n}) + \frac{p_1^4}{4}(b_2 + \tau\frac{b_1^2}{2^n})^2\right) \\
& + \frac{\tau b_1p_1}{(1+\tau)}\left(b_1(p_3 - p_1p_2 + \frac{p_1^3}{4}) + (b_2p_1 + \frac{3b_1^2p_1}{2})(p_2 - \frac{p_1^2}{2}) + \frac{3\tau p_1^2}{2^{1-n}(1+\tau)}(b_2 + \tau\frac{b_1^2}{2^n})^2\right) \\
& - \frac{5\tau}{2^{2-n}(1+\tau)}\left(b_1^3p_1^2(p_2 - \frac{p_1^2}{2}) + \frac{b_1p_1^3}{2}(b_2 + \tau\frac{b_1^2}{2^n})\right) + \frac{\tau b_1^4p_1^4}{2^4}\Bigg| \qquad (32)
\end{aligned}
$$

Applying Lemma 4 (when$\mu = 1$), Lemma 2 and Lemma 1 results to

$$
\begin{aligned}
|a_5| \leq & \frac{2b_1 - 3b_2 + 9b_3}{5^nC} + \frac{(A-1)[4b_1^2 + 8Db_1^2 + 4D^2]}{2\cdot 3^n\cdot 5^nA^2C} + \frac{2(A-1)b_1}{5^nBC}\left[2b_1 + 4\left(b_2 + \frac{3}{2}b_1^2\right) + \frac{3(A-1)\mathrm{D}}{2^{-(1+n)}}\right] \\
& + \frac{(A-1)b_1^4}{5^nC} + \frac{5(A-1)[2b_1^3 + 4Db_1]}{2^{2-n}5^nAC}, \qquad (33)
\end{aligned}
$$

where $A = (1+\tau)$, $B = (1+2\tau)$, $C = (1+3\tau)$ and $D = b_1 + (A-1)\frac{b_1^2}{2^n}$. ☐

Theorem 2. Let $f \in A_n^\tau(\beta)$, then

$$
FS(\sigma, f) = |a_3 - \sigma a_2^2| \leq \begin{cases} \dfrac{2^nb_2 + \tau b_1^2 - \sigma 2^{-n}3^n(1+\tau)b_1^2}{2^n * 3^n(1+\tau)}, & \text{if } \sigma \leq T_1 \\ \dfrac{b_1}{3^n(1+\tau)}, & \text{if } T_1 \leq \sigma \leq T_2 \\ \dfrac{\sigma 2^{-n}3^n(1+\tau)b_1^2 - 2^nb_2 - \tau b_1^2}{2^n * 3^n(1+\tau)}, & \text{if } \sigma \geq T_2 \\ 2\max[1, T_3], & \text{if } \sigma \in \mathbb{C} \end{cases} \qquad (34)
$$

where

$$
T_1 = \frac{2^n((\tau b_1 - 2^n)b_1 + 2^nb_2)}{3^n(1+\tau)b_1^2}, \quad T_2 = \frac{2^n((\tau b_1 + 2^n)b_1 + 2^nb_2)}{3^n(1+\tau)b_1^2}, \quad T_3 = \frac{\sigma 2^{-n}3^n(1+\tau)b_1^2 - \tau b_1^2 - 2^nb_2}{2^{n+1}\cdot 3^n(1+\tau)}.
$$

Proof. From (24) and (23),

$$a_3 - \sigma a_2^2 = \frac{b_1}{2 \cdot 3^n(1+\tau)}\left(p_2 - \frac{p_1^2}{2}\left(\frac{2^n b_1 - 2^n b_2 - \tau b_1^2 + \sigma 2^{-n}3^n(1+\tau)b_1^2}{2^n b_1}\right)\right) \quad (35)$$

Which implies that,

$$|a_3 - \sigma a_2^2| = \frac{b_1}{2 \cdot 3^n(1+\tau)}\left|\left(p_2 - \frac{p_1^2}{2}\left(\frac{2^n b_1 - 2^n b_2 - \tau b_1^2 + \sigma 2^{-n}3^n(1+\tau)b_1^2}{2^n b_1}\right)\right)\right|.$$

From (35), let

$$\Lambda = \frac{2^n b_1 - 2^n b_2 - \tau b_1^2 + \sigma 2^{-n}3^n(1+\tau)b_1^2}{2^n b_1}$$

Applying Lemma 2 on 35 so that,

$$\left|p_2 - \Lambda\frac{p_1^2}{2}\right| \leq 2(1-\Lambda) = \frac{2^n b_2 + \tau b_1^2 - \sigma 2^{-n}3^n(1+\tau)b_1^2}{2^{n-1} * b_1} \quad (36)$$

If $\Lambda \leq 0$, we have

$$\sigma \leq \frac{2^n((\tau b_1 - 2^n)b_1 + 2^n b_2)}{3^n(1+\tau)b_1^2}. \quad (37)$$

Also,

$$\left|p_2 - \Lambda\frac{p_1^2}{2}\right| \leq 2, \quad (38)$$

if $0 \leq \Lambda \leq 2$, we obtain

$$\frac{2^n((\tau b_1 - 2^n)b_1 + 2^n b_2)}{3^n(1+\tau)b_1^2} \leq \sigma \leq \frac{2^n((\tau b_1 + 2^n)b_1 + 2^n b_2)}{3^n(1+\tau)b_1^2}. \quad (39)$$

Furthermore,

$$\left|p_2 - \Lambda\frac{p_1^2}{2}\right| \leq 2(\Lambda - 1) = \frac{\sigma 2^{-n}3^n(1+\tau)b_1^2 - 2^n b_2 - \tau b_1^2}{2^{n-1}b_1}, \quad (40)$$

if $\Lambda \geq 2$, we have

$$\sigma \geq \frac{2^n((\tau b_1 + 2^n)b_1 + 2^n b_2)}{3^n(1+\tau)b_1^2}. \tag{41}$$

Finally,

$$\left|p_2 - \Lambda\frac{p_1^2}{2}\right| \leq 2\max(1, |1-\Lambda|) = 2\max\left(1, \frac{\sigma 2^{-n}3^n(1+\tau)b_1^2 - \tau b_1^2 - 2^n b_2}{2^n b_1}\right), \tag{42}$$

if $\Lambda \in \mathbb{C}$. Substituting (36) – (42) into (35) gives the desired results. ☐

Conclusion

This study established upper estimate for coefficient bounds and the Fekete-Szegö functional for functions in $A_n^\tau(\beta)$, defined via subordination using Sälägean differential operator associated with Ma-Minda function. The subclass generalises some well known subclasses of univalent functions. By varying the parameters our results recover those of existing subclasses like Yamaguchi and Star-like functions. The work done here may be further investigated using more general operators to extend the class or study applications in related fields such as complex dynamic systems.

Recommendations

This work explores rich areas of geometric function theory, where the study of coefficient bounds and Fekete-Szego functional continues to yield deep insights into structure and behaviour of analytic-univalent functions, these properties are applicable in engineering for control systems and mechanical designs, etc.

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